Riemannian metrics

Def. Let M^n be a smooth manifold. A <u>riemannian metric</u> on M^n is a smooth mapping $g : T(M) \times_M T(M) \longrightarrow \mathbb{R}$ such that for each $p \in M$ the restricted map $g_p : T_p(M) \times T_p(M) \longrightarrow \mathbb{R}$ is an inner product on the vector space $T_p(M)$.

EXAMPLES.

0. On $\mathbb{R}^n$, take the map

$$
\begin{array}{ccc}
g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n & \longrightarrow & \mathbb{R} \\
(x,y,z) \Big\downarrow \text{coord projection} & & \\
\text{proj. map} \downarrow \;\; x \quad \mathbb{R}^n & &
\end{array}
$$

$g(x; y, z) = \langle y, z \rangle$ the standard inner product.

$\boxed{\text{standard metric on } \mathbb{R}^n}$

1. Let G be an $n \times n$ matrix which is symmetric and positive definite (${}^T x G x > 0$ if $x \neq 0$).
Then $g(x; y, z) = {}^T y G z$ is also a riemann metric.
More generally, this holds if $G : \mathbb{R}^n \longrightarrow n \times n$ pos. def. matrices is a smooth function

$$G(x) = \begin{pmatrix} g_{11}(x) & \dots & g_{1n}(x) \\ & \vdots & \\ g_{n1}(x) & \dots & g_{nn}(x) \end{pmatrix}$$ where $g_{ij} = g_{ji}$

* equiv to pos. def.

and the principal minors are all positive.*

One can also extend to metrics on U open in $\mathbb{R}^n$.

2. Induced metric on a submanifold

Suppose $f : N^k \to M^n$ is a smooth embedding, so that $T(f) : T(N) \to T(M)$ also is a smooth embedding which maps each $T_p(N)$ to $T_{f(p)}(M)$ linearly. The methods of Ch. 3 imply that there is a smooth map

$$T^{(2)}(f) : T(N) \times_N T(N) \longrightarrow T(M) \times T(M)$$

which locally looks like

$$\begin{array}{ccc} U \times \mathbb{R}^k \times \mathbb{R}^k & \longrightarrow & U \times N_\delta(0; \mathbb{R}^{n-k}) \times \mathbb{R}^n \times \mathbb{R}^n \\ (u; y, z) & \longrightarrow & (u, 0; j(y), j(z)) \end{array}$$

where $j : \mathbb{R}^k \longrightarrow \mathbb{R}^n \cong \mathbb{R}^k \times \mathbb{R}^{n-k}$, $j(x) = (x, 0)$.

Special cases

$f : N \longrightarrow \mathbb{R}^n$ smooth embedding

$f : S^{n-1} \longrightarrow \mathbb{R}^n$ standard inclusion

EXISTENCE THEOREM. Every second countable smooth manifold M^n supports a riemannian metric.

Sketch of proof. Take a countable locally finite smooth atlas $h_\alpha : W_\alpha \longrightarrow N_3(0;\mathbb{R}^n)$ such that the sets $h_\alpha^{-1}[N_1(0;\mathbb{R}^n)]$ still cover N. Let g_α be a riemannian metric on W_α (which exists because $W_\alpha \cong N_3(0;\mathbb{R}^n)$ and the latter has many riemannian metrics). Let $\varphi : N_3(0;\mathbb{R}^n) \to \mathbb{R}$ be a smooth bump function which is 1 if $|x| \leq 1$ and 0 if $|x| \geq 2$.

Extend $\varphi h_\alpha \cdot g_\alpha$ to a smooth map on $T_p(M) \times_M T(M)$ by setting it equal to zero off $T(W_\alpha) \times_{W_\alpha} T(W_\alpha)$. The resulting map $\overline{g_\alpha} : T(M) \times_M T(M) \longrightarrow \mathbb{R}$ has all but one of the required properties for a riemannian metric: We always have $\overline{g_\alpha}(v,v) \geq 0$, but $\overline{g_\alpha}(v,v) = 0$ does not necessarily imply $v = 0$. However, if v is a tangent vector over a point in $h_\alpha^{-1}[N_1(0,\mathbb{R}^n)]$, then $\overline{g_\alpha}(v,v) = 0$ does imply $v = 0$.

Now form the sum $g = \sum \overline{g_\alpha}$. This also has the properties of a riemannian metric except possibly $g(v,v) = 0 \Rightarrow v = 0$. At any rate we have $g(v,v) \geq 0$. But now v lies over a point in <u>some</u> $h_{\alpha_0}^{-1}[N_1(0,\mathbb{R}^n)]$, so that $\overline{g_{\alpha_0}}(v,v) > 0$. Since $\overline{g_{\alpha_0}}$ is a summand of g, this yields $g(v,v) > 0$ and hence g is in fact a riemannian metric. ∎

EXAMPLE 3. The riemannian metric for hyperbolic geometry

$$M^n = N_1(0;\mathbb{R}^n)$$

$$g(x;y,z) = \frac{\langle y,z\rangle}{1-|x|^2}.$$

<u>Distance function associated to a riemannian metric.</u> Suppose M^n is connected and has a riemannian metric g. Let γ be a piecewise smooth curve $[a,b] \longrightarrow M$. Then the tangent vectors $\gamma' : [a,b] \longrightarrow T(M)$ yield a map which is continuous at all but finitely many points, so we can define a length

$$L_\gamma = \int_a^b \sqrt{g(\gamma'(t),\gamma'(t))}\,dt.$$

<u>Def.</u> Given this setting and points $p, q \in M$, let $d_g(p,q) = $ G.L.B. L_γ, where γ runs through all piecewise smooth curves joining p to q.

The following result is not particularly hard to prove, but we shall not do so here:

THEOREM. The function d^g defines a metric on M, and the metric topology is the original topology on M.

EXAMPLE Let $M^n = \mathbb{R}^n - \{0\}$ with the usual riemannian metric, and let $0 \neq v$. Then the distance from v to $-v$ is $2|v|$ but there is no smooth curve in M^n which joins them and has length $2|v|$.

Finally, we have a remarkable converse to the examples at the top of page 5C-3:

THEOREM OF J. F. NASH. If M^n is a smooth manifold (connected) and g is a riemannian metric on M^n, then there is a smooth embedding $f: M \to \mathbb{R}^q$ (some q) such that g is the riemannian metric induced from f and the standard metric on $\mathbb{R}^q$.