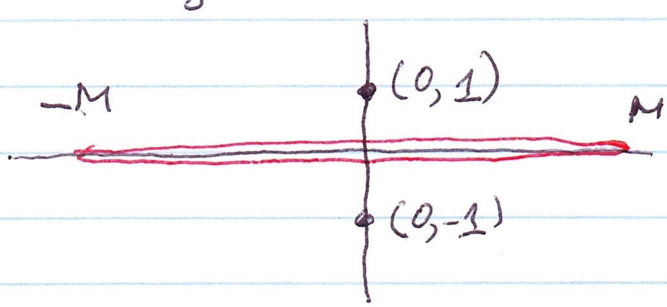
More on riemannian distance

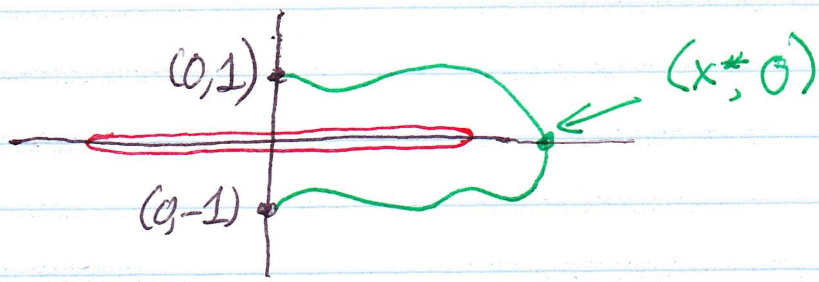
1. Chapter 13 of Lee contains a proof of the result stating that riemannian distance yields a metric for the topology on M (associated to the riemannian metric g).

2. As noted in Chapter 13 of Lee, if U is open in $\mathbb{R}^n$ and g is the standard metric on $\mathbb{R}^n$, then the riemannian distance $d^{g,U}(x,y) \geq |x-y|$ (the usual distance). However, the first may be much larger than the second. Consider the following example: $U = \mathbb{R}^2 - [-M,M] \times \{0\}$



The Euclidean distance between $(0,-1)$ and $(0,1)$ is 2, but we claim that the riemannian distance is at least $2\sqrt{1+M^2}$.

Verification Let γ be a rectifiable continuous curve in U joining $(0,-1)$ to $(0,1)$. By connectedness, γ must cross the x-axis at some point $(x^*,0) \in U$, and we must have $|x^*| > M$.



Clearly, $\text{length}(\gamma) \geq d((0,-1),(x^*,0)) + d((x^*,0),(0,1))$ (d = Euclidean distance).

Each term in the sum is equal to $\sqrt{1+(x^*)^2}$, so $\text{length}(\gamma) \geq 2\sqrt{1+x^2}$ for some x s.t. $|x| > M$. Therefore $d^{g,U}((0,-1),(0,-1)) \geq 2\sqrt{1+M^2}$.

In fact, if we take the broken line curves joining $(0,-1)$ to $(x,0)$ and $(x,0)$ to $(0,1)$ for an arbitrary $x > M$, we see that the riemannian distance from $(0,-1)$ to $(0,1)$ is exactly $2\sqrt{1+M^2}$.