Surfaces in $\mathbb{R}^3$

The purpose of this section is to indicate how the machinery developed in this course relates to the classical differential geometry of surfaces in 3-space.

Let $M^2 \subseteq \mathbb{R}^3$ be a smooth 2-manifold (= surface).

Def. A (classical) orientation of M^2 is a smooth function $N : M \longrightarrow S^2$ such that $|N(p)| = 1$ for all p and $N(p) \perp T_p(M)$ for all $p \in M$.

[unit normal vector field on M]

EXAMPLES

1. $S^2 \subseteq \mathbb{R}^3$, with $N(p) = p$ (outward normal) or $N(p) = -p$ (inward normal)

2. More generally, if $U \subseteq \mathbb{R}^3$ is open and 0 is a regular value of $f : U \longrightarrow \mathbb{R}^3$, then we can take $N(p) = |\nabla f(p)|^{-1} \cdot \nabla f(p)$ or its negative.

3. Suppose M is the image of a smooth embedding $\sigma : U \to \mathbb{R}^3$ where U is open and connected. Then we can define N by

$$\frac{1}{\left|\frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v}\right|} \cdot \left(\frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v}\right)$$

or its negative. Furthermore, if $N' : U \to \mathbb{R}^3$ is a unit normal vector field, then either N' is given by the expression above or by its negative.

(For the second part, at each $x \in U$ we have $N'(x) = N(x)$ or $N'(x) = -N(x)$, so $N'(x) = \varepsilon(x) N(x)$ where $\varepsilon(x) = \pm 1$. By continuity, ε is locally constant, so by connectedness it is constant.)

4. The Möbius strip is a standard example of a surface which does not support an orientation. Here is one way of proving this:

We can view the Möbius strip as the image of the immersion

$\varphi : \mathbb{R} \times (-\frac{1}{2}, \frac{1}{2}) \longrightarrow \mathbb{R}^3$ defined by

$$\varphi(u, v) = (\cos u, \sin u, 0) + v\left(0, \cos\frac{u}{2}, \sin\frac{u}{2}\right).$$

[Details of the verification will be omitted.]

Furthermore, the inverses of φ restricted to

$V_k = \left((k-1)\frac{\pi}{2}, (k+1)\frac{\pi}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ form a smooth atlas whose pieces are connected open subsets, where $k = 1, \dots, 4$.

Suppose now that we have an orientation N, and let N_k be the pulled back restriction over V_k. Then by connectedness we have

$$N_k = \varepsilon_k \frac{1}{\left|\frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v}\right|} \cdot \left(\frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v}\right) \text{ for some}$$

ε_k's $= \pm 1$. Without loss of generality we can assume $\varepsilon_1 = +1$ (take negatives to get the other case).

By construction $N_1 | V_1 \cap V_2 = N_2 | V_1 \cap V_2$, so it follows that $\varepsilon_1 = \varepsilon_2$. Similarly, we see that $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4$, which implies that the values of

$$\frac{1}{\left|\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}\right|} \left(\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}\right)$$

are the same for (u, v) and $(u + 2\pi, v)$ if $u \in \left(0, \frac{\pi}{2}\right)$. However, one can check directly that the value at the second point is the negative of the value at the first. — The source of this contradiction is our assumption that N exists, so there cannot be an N with the required properties. ∎

The Second Fundamental Form.

The first fundamental form in classical surface theory corresponds to the riemannian metric on $M \subseteq \mathbb{R}^3$ induced by the standard one on $\mathbb{R}^3$.

Explicitly, if U is open in $\mathbb{R}^2$ and $\sigma : U \to M$ is a diffeomorphism onto a neighborhood of $p \in M$, then the coefficients in the classical expression

$$\mathrm{I} = E\,du\,du + 2F\,du\,dv + G\,dv\,dv$$

are given by the Gram matrix entries:

$$E = \left\langle \frac{\partial\sigma}{\partial u}, \frac{\partial\sigma}{\partial u} \right\rangle, \quad F = \left\langle \frac{\partial\sigma}{\partial u}, \frac{\partial\sigma}{\partial v} \right\rangle, \quad G = \left\langle \frac{\partial\sigma}{\partial v}, \frac{\partial\sigma}{\partial v} \right\rangle.$$

Now suppose that M is oriented with unit normal $\vec{N}$, and suppose that $p = \sigma(x)$.

KEY FACT. If $A \in T_p(M) \subseteq \{p\} \times \mathbb{R}^3$, then $D\vec{N}(p)[A] \in T_p(M)$.

This is a restatement of the fact that

$$|\vec{N}|^2 = 1 \Rightarrow \frac{\partial \vec{N}}{\partial u} \cdot \vec{N} = 0 = \frac{\partial \vec{N}}{\partial v} \cdot \vec{N}. \blacksquare$$

It follows that $D\vec{N}$ defines a smooth map

$$\begin{array}{ccc} T(M) & \xrightarrow{\Phi} & T(M) \\ & {\scriptstyle \tau_M} \searrow \quad \swarrow {\scriptstyle \tau_M} & \\ & M & \end{array}$$

such that for each $p \in M$, Φ maps $T_p(M)$ to itself linearly.

In the book by Hicks, Φ is called the $\boxed{\text{Weingarten map}}$, and in the book by O'Neill the $\boxed{\text{negative}}$ of this map is called the $\boxed{\text{Shape Operator.}}$

(As O'Neill notes, the sign choice simplifies some other statements.)

<u>Def.</u> The <u>second fundamental form</u> is the map $\mathrm{II}: T(M) \times_M T(M) \longrightarrow \mathbb{R}$ given by $\mathrm{II}(A, B) = \langle -\Phi(A), B \rangle$.

note!

<u>ANOTHER KEY FACT.</u> The mapping Φ is <u>self-adjoint</u>: $\langle \Phi(A), B \rangle = \langle B, \Phi(A) \rangle$ for all $A, B \in T_p(M)$ and all $p \in M$.

Here are some references:

<u>CLASSICAL</u>

S. Lipschutz, Schaum's Outline of Theory and Problems in Differential Geometry

MODERN

N. Hicks, Notes on Differential Geometry

B. O'Neill, Elementary Differential Geometry (Second Revised Edition).

this differs from the Second Edition!!

In the classical approach to the subject, the second fundamental form is expressed as $\mathrm{II} = e\,du\,du + 2f\,du\,dv + g\,dv\,dv$,

where
$$e = \left\langle \Phi\left(\frac{\partial\sigma}{\partial u}\right), \frac{\partial\sigma}{\partial u} \right\rangle$$

(often other letters are used instead)

$$f = \left\langle \Phi\left(\frac{\partial\sigma}{\partial v}\right), \frac{\partial\sigma}{\partial u} \right\rangle = \left\langle \Phi\left(\frac{\partial\sigma}{\partial u}\right), \frac{\partial\sigma}{\partial v} \right\rangle$$

$$g = \left\langle \Phi\left(\frac{\partial\sigma}{\partial v}\right), \frac{\partial\sigma}{\partial v} \right\rangle.$$

Most of the time the second fundamental form is $\boxed{\text{not}}$ an inner product. For example, if $M \subseteq \mathbb{R}^3$ is the xy plane then $\vec{N}$ is constant and hence $\mathrm{II}(A, B)$ is always zero.

A fundamental result in surface theory implies that a surface is completely determined locally by

1. its fundamental forms (both),
2. an initial choice of a point p and a unit vector $\vec{N}_0$.

Proofs are given in Lipschutz and Hicks.

See dgnotes2012.pdf for more on the classical differential geometry of curves and surfaces.