6. Approximation and embedding theorems

Frequently it is useful — and sometimes it is absolutely crucial — to approximate a continuous map between smooth manifolds by another map with better properties. For example, if $P \to M$ and $Q \to M$ are smooth embeddings, it is often useful to approximate the first embedding by another one so that the approximated image of P intersects Q transversely.

SMOOTH APPROXIMATION THEOREM. Let $f: M^m \to N^n$ be continuous, where M^m and N^n are smooth manifolds and N^n has a metric d_N, and let $\delta: M \to (0, \infty)$ be continuous. Then there is a smooth mapping $g: M \to N$ which is a δ-approximation to f; i.e.,

$$d_N(f(x), g(x)) < \delta(x) \text{ for all } x \in M.$$

Later in this chapter we shall prove that N is metrizable, so the result applies to all $f: M \to N$.

(Recall that manifolds are 2nd countable.)

Proof. Start with a countable open covering of N^n by smooth charts $W_\alpha \to N_3(0;\mathbb{R}^n)$. Next, take a countable locally finite open covering of M^m by smooth charts $\Omega_j \to N_3(0;\mathbb{R}^m)$ such that f maps each Ω_j into some W_α. Let Ω_j' and Ω_j'' denote the subdisks of radius 1 and 2 respectively. Let $\varphi_j : M \to [0,1]$ be a smooth bump function such that $\varphi_j = 1$ on $\overline{\Omega_j'}$ and $\varphi_j = 0$ on $M - \Omega_j''$. Define a sequence of compact subspaces $K_j \subseteq M$ by $K_0 = \phi$ and $K_j = K_{j-1} \cup \overline{\Omega_j'}$.

CLAIM: Assume the charts are chosen so that the sets Ω_j' already cover M^m. — We can find a sequence of functions $f_j : M \to N$ such that the following hold:

(1) f_j is smooth on $K_j \cdot \Omega_1' \cup \ldots \cup \Omega_j'$

(2) $f_j = f_{j-1}$ off $M - \Omega_j''$.

(3) f_j is a $\delta/2^{j+1}$ approximation to f_{j-1}.

If the claim is true, then ~~g~~ $g = \lim_{j\to\infty} f_j$ will be the desired smooth δ-approximation to f.

Start with $f_0 = f_g$ and assume f_{j-1} has been constructed. We are only going to modify the map on Ω_j, so we shall focus on the local model $\Omega_j \longrightarrow N_3(0; \mathbb{R}^n)$ for f_{j-1}. Let $C \subseteq \Omega_j$ be the compact set corresponding to the intersection of K_{j-1} with $\overline{\Omega_j''}$, the closed disk of radius 2. By the Stone-Weierstrass approx thm., we can uniformly approximate f_{j-1} by a polynomial function (in n variables) on $\overline{\Omega_j''}$. If we approximate closely enough, we can find a polynomial h such that $f_{j-1} + h$ maps $\overline{\Omega_j''}$ into $N_3(0; \mathbb{R}^n)$ and is a $\delta/2^{j+1}$ appx. to f_j. In fact we can insist on the latter for $f_{j-1} + t \cdot h$ for all t such that $0 \leq t \leq 1$. Now define f_j by $(1-\varphi) f_{j-1} + \varphi \cdot h$. It is routine to check that f_j has the required properties. ■

COMPLEMENT. The smooth approximation is in fact homotopic to f.

A slightly more delicate argument proves the following stronger result:

RELATIVE APPROXIMATION THEOREM. Suppose that $f : M^m \longrightarrow N^n$ is continuous and f is smooth on an open neighborhood of a closed subset E. Then one can construct the approximation so that $g = f$ on a slightly smaller open neighborhood.

Sketch of the proof. Let Ω be an open nbhd of E, and let $W = M - \Omega$. Construct an open covering as in the proof of the absolute theorem, with the added condition that each open subset from the covering is contained in either Ω or W. Now index the covering by integers so that the sets with nonpositive indices are contained in Ω. The approximation is then constructed recursively as before. ■

Note that this proof is different from the one in Lee.