Sard's Theorem

Def. M smooth manifold, $E \subseteq M$. Say E has measure zero if for every smooth chart $h_\alpha : W_\alpha \to U_\alpha \subseteq \mathbb{R}^n$ the set $h_\alpha[W_\alpha \cap E]$ has measure zero in the usual sense.

One can define "reasonable" measures on M such that the definition above corresponds to sets of measure zero for such measures, but we do not need such extra structure.

THEOREM A. $E \subseteq M$ has measure $0 \iff$ for all smooth charts in some compatible countable smooth atlas the sets $h_\alpha[W_\alpha \cap E]$ all have measure zero in the usual sense.

This will be deduced from the following local result:

THEOREM B. Let U, V be open in $\mathbb{R}^n$ and let $f : U \to V$ be smooth. If $E \subseteq U$ has Lebesgue measure equal to zero, then so does $f[E]$.

<u>Proof of Theorem B.</u> Take a countable open covering of U by open disks U_k such that the subdisks V_k with $1/3$ the radii still cover U. Let W_k be the subdisks with $2/3$ the radii. It will suffice to prove that each set $E \cap \overline{V_k}$ has measure zero (since a countable union of sets with measure zero also has measure zero).

<u>CLAIM:</u> $f|\overline{W_k}$ satisfies a <u>Lipschitz condition</u>: There is a constant $C_k > 0$ such that $|f(u) - f(v)| \leq C_k \cdot |u - v|$ for all $u, v \in \overline{W_k}$.

<u>PROOF OF CLAIM:</u> By compactness of $\overline{W_k}$ and the smoothness of f, the coefficients (entries) of $Df(y)$ are bounded on $\overline{W_k}$. By a version of the fundamental thm. of calculus, we have

$$f(u) - f(v) = \int_0^1 Df(tu + (1-t)v)[u - v]\, dt,$$ which

implies $|f(u) - f(v)| \leq \int_0^1 \text{Const.} \cdot |u - v|\, dt = \text{Const.} \cdot |u - v|$. ∎

Proof of Theorem B resumed.

Let $\varepsilon > 0$. Since $E \cap \overline{V_k}$ has measure zero, there is a countable union of disks $D_1, D_2, \ldots$ such that $D_i \subseteq W_k$ (all i), and $E \cap \overline{V_k} \subseteq \bigcup_i D_i$ and $\sum_i \text{meas}(D_i) < \frac{\varepsilon}{C_k^n}$. By the Lipschitz condition, f maps each D_i into a disk D_i' with radius $\leq C_k \cdot \text{radius}\, D_i$. Therefore $f[E \cap \overline{V_k}] \subseteq \bigcup_i D_i'$, and $\sum_i \text{meas}(D_i') < \varepsilon$. But this means that $E \cap \overline{V_k}$ has Lebesgue measure zero. As noted above, this suffices to prove that E has Lebesgue measure zero. ∎

Proof of Theorem A. Let $\{h_j : W_j \to U_j\}$ be the given smooth atlas for M, and let $k : \mathcal{S}_0 \to V$ be a compatible chart. We need to prove that $k[E \cap \mathcal{S}_0]$ has Lebesgue measure zero.

Now $\Omega = \cup_j (\Omega \cap W_j)$, so $E \cap \Omega = \cup_j (E \cap \Omega \cap W_j)$, so it suffices to prove that each set $k[E \cap \Omega \cap W_j]$ has measure zero. Since E has measure zero, the same is true for each set $h_j^{-1}[E \cap \Omega \cap W_j]$ by the hypotheses and the fact that $E' \subseteq E$ & $\mu E = 0 \Rightarrow \mu E' = 0$, where μ = Lebesgue meas. By the assumptions the transition map $k h_j^{-1}$ is smooth, so by Theorem B we see that $\mu(h_j^{-1}[E \cap \Omega \cap W_j]) = 0 \Rightarrow \mu(k[E \cap \Omega \cap W_j]) = 0$ for all j. Therefore $\mu(k[E \cap \Omega]) = 0$ also. ∎

Easy Case of Sard's Theorem.

Let $f : M^m \longrightarrow N^n$ be smooth, where $m < n$. Then $f[M^m]$ has measure zero.

HARD CASE (in Lee) Say that $y \in N$ is a singular value of N w.r.t. $f: M \to N$ if there is some $x \in M$ such that $f(x) = y$ and $T(f)_x : T_x(M) \longrightarrow T_y(N)$ is **NOT** onto. Then the set of singular values has measure zero.

Proof of the easy case of Sard's Theorem.

Theorem B generalizes immediately to the following result: If $f: N_1^n \longrightarrow N_2^n$ is smooth and $E \subseteq N_1^n$ has measure zero, then so does $f[E] \subseteq N_2$.

Suppose now we are given $f: M^m \longrightarrow N^n$ where $m < n$. Then $M \times \{0\} \subseteq M \times \mathbb{R}^{n-m}$ has measure zero (verify this!). Consider the map

$$F: M \times \mathbb{R}^{n-m} \xrightarrow[\text{proj}]{} M \xrightarrow{f} N.$$ Then

$F[M \times \{0\}]$ has measure zero. Since $F[M \times \{0\}] = f[M]$, it follows that $f[M]$ has measure zero. ∎