Smooth embeddings in $\mathbb{R}^k$

"EASY" WHITNEY EMBEDDING THEOREM. If M^n is a (2nd countable) smooth manifold, then there is a closed smooth embedding $f: M^n \longrightarrow \mathbb{R}^{2n+1}$.

The Hard Whitney Embedding Thm. states that $\mathbb{R}^{2n+1}$ can be replaced by $\mathbb{R}^{2n}$. However, there are examples where M^n does not even embed topologically in $\mathbb{R}^{2n-1}$.

There is a clear and uncluttered proof of this result in Milnor's 1958 Differential Topology note (which are in the course directory). See pp. 5-7 (1.18-1.20) and pp. 8-10 (1.27-1.32). The key steps are two approximation theorems.

FIRST APPROXIMATION. Every smooth $f: M^n \longrightarrow \mathbb{R}^k$ can be approximated by an immersion if $k \geq 2n$.

SECOND APPROXIMATION. Every smooth immersion $f : M^n \longrightarrow \mathbb{R}^k$ can be approximated by a 1-1 immersion if $k \geq 2n+1$.

Some comments on the proofs appear below.

DERIVATION OF WHITNEY'S THEOREM.

Start with a proper smooth map $f : M^n \longrightarrow \mathbb{R}$, and approximate f by a 1-1 immersion $g : M^n \longrightarrow \mathbb{R}^{2n+1}$ (strictly speaking, approximate $M \longrightarrow \mathbb{R} \cong \mathbb{R} \times \{0\} \subseteq \mathbb{R}^{2n+1}$). Choose $\delta : M^n \longrightarrow \mathbb{R}$ to be $\delta(x) \equiv 1$. If we can show that g is a closed map, we are done. As in proper.pdf, this reduces to showing f is proper. However, this is easy to check because f proper and $|f(x) - g(x)| < 1$ all x implies g is proper (verify this!). ∎

Note that Milnor's argument only uses the easy case of Sard's Theorem.

Note on the proof of the Second Approximation

On the 7^{th} line from the bottom of p. 10, there is an assertion, "It follows that $f_k(y) = f_k(y_0) \Longleftrightarrow \varphi_k(y) = \varphi_k(y_0)$ and $f_{k-1}(y) = f_{k-1}(y_0)$."

Here are more details:

The ($\Longleftarrow$) direction is obvious. Suppose now that $f_k(y) = f_k(y_0)$ but $\varphi_k(y) \neq \varphi_k(y_0)$. Then we have

$$\frac{f_k(y) - f_k(y_0)}{\varphi_k(y) - \varphi_k(y_0)}$$ contradicting the choice of b_k.

So $f_k(y) = f_k(y_0) \Rightarrow \varphi_k(y) = \varphi_k(y_0)$. But then

$$f_{k-1}(y) = f_k(y) - \varphi_k(y) b_k = f_{k-1}(y_0) - \varphi_k(y_0) b_k = f_k(y_0),$$

so the assertion is verified.

Proceeding to the next paragraph, if $g = \lim f_k$ and $g(y) = g(y_0)$, then $f_k(y) = f_k(y_0)$ for all sufficiently large k. As in Milnor's notes this leads to $f(y) = f(y_0)$; and $\varphi_k(y) = \varphi_k(y_0)$ for all k.

$f(y) = f(y_0) \Rightarrow$ y and y_0 cannot belong to a common coordinate neighborhood U_i.

However $\varphi_k(y) = \varphi_k(y_0)$ all $k \Rightarrow$ when one is positive so is the other. Since some $\varphi_j(y) > 0$, it follows that some U_j contains both y and y_0. This is a contradiction since $f|U_j$ is 1-1 on U_j; the source of the problem is the assumption that $g(y) = g(y_0)$ but $y \neq y_0$. Therefore g is 1-1. ■