7. Lie groups

Hierarchy of definitions

A $\boxed{\text{group}}$ can be defined as a (nonempty) set G together with functions $m: G\times G \to G$, $u: \{1\} \longrightarrow G$, $i: G \longrightarrow G$ which satisfy the standard identities for multiplication, unit element and inverse respectively.

A $\boxed{\text{topological group}}$ is a topological space G with a group structure s.t. m and i are continuous (u will automatically be continuous).

A $\boxed{\text{Lie group}}$ (pronounced LEE) is a smooth manifold G with a group structure s.t. m and i are smooth (u will automatically be smooth).

So Lie group $\Rightarrow$ topological group $\Rightarrow$ group. We shall work backwards, first discussing topological groups and then passing to Lie groups.

General topology texts usually have sets of exercises which develop the most basic properties of topological groups. For example, there are supplementary exercises at the end of Chapter 2 in Munkres, Topology (Second Edition). There is a more detailed treatment in Appendix A from the document gentop-notes.pdf.

Passage to Lie groups One noteworthy result about (Hausdorff) topological groups is that if G is a connected topological group, then every open neighborhood of the identity generates G. A strengthening of this principle has extremely far-reaching implications for Lie groups.

Comment on Lee, p. 150 There is an assertion that, at the time of Lie's work, "global topological notions such as manifolds ... had not been formulated."

Two points should be mentioned:

1. At the time, the concept of a manifold was vaguely understodd thanks to the ideas of Riemann.

2. Many of the local groups that Lie studied cannot be globalized.

Locally compact groups The previously cited Appendix A notes that such groups have a canonical measure space structure. In fact, the theory of Hausdorff locally compact Lie groups is well understood. One main consequence is that if a topological group G is also a topological manifold, then it is topologically isomorphic to a Lie group. This result, proved in the 1950s by A. Gleason and independently by D. Montgomery and L. Zippin, is often known as the solution to Hilbert's Fifth Problem.

Some constructions on Lie groups

Direct products. If G and H are Lie groups, then the direct product $G \times H$ is a Lie group with respect to the direct product group structure and the product smooth structure.

Semidirect products. If G and H are groups and $\theta : G \to \mathrm{Aut}(H)$ is a homomorphism, then the semidirect product $H \rtimes_\theta G$ is given by

$G \times H$ with multiplication given by

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2)$$

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$$(h_1, g_1) \cdot (h_2, g_2) = (h_1 \theta(g_1) h_2, g_1 g_2).$$

This group is characterized by the following axioms:

(1) $H \triangleleft \Gamma$ and $G \cong \Gamma / H$

(2) The action of G on H by inner automorphisms is given by θ.

(3) There is a homomorphism $G \to \Gamma$ such that $G \to \Gamma \to G$ is onto.

Suppose now that G is a Lie group, $H \cong \mathbb{R}^n$, and $\theta : G \to GL(n, \mathbb{R})$ is a smooth homomorphism. Then the semidirect product is a Lie group, where one takes the product smooth structure on $\mathbb{R}^n \times G$.

<u>Example</u> The <u>affine group</u> $Aff(\mathbb{R}^n)$ is obtained by taking $G = GL(n, \mathbb{R})$, with $\theta =$ identity. It is isomorphic to the group of transformations on $\mathbb{R}^n$ having the form $T(x) = Ax + b$ where $A \in GL(n, \mathbb{R})$ and $b \in \mathbb{R}^n$.

<u>Universal coverings.</u> If G is a connected Lie group, and $\tilde{G} \to G$ is a universal covering, then one can lift the unit map to $\tilde{u} : \{1\} \to \tilde{G}$, the multiplication m to $\tilde{m} : \tilde{G} \times \tilde{G} \to \tilde{G}$ and the inverse i to $\tilde{i} : \tilde{G} \to \tilde{G}$ such that $\tilde{G}$ becomes a Lie group and the covering space projection $p : \tilde{G} \to G$ is a smooth homomorphism.

Note that if $K = \text{Kernel } \widetilde{G} \to G$, then K is a discrete normal subgroup of $\widetilde{G}$, so by a general result on topological groups K is contained in the center of $\widetilde{G}$.

Elementary but important facts.

Let G be a Lie group, and let $a \in G$. Then the $\left\{ \begin{array}{l} \text{left multiplication map } L_a(x) = ax \\ \text{right multiplication map } R_a(x) = xa \end{array} \right\}$ are diffeomorphisms.

These maps have the following properties:

$$L_{ab} = L_a \circ L_b \quad R_{ba} = R_a \circ R_b$$

$$L_{a^{-1}} = (L_a)^{-1} \quad R_{a^{-1}} = (R_a)^{-1}$$

$$L_1 = R_1 = \text{identity on } G.$$

(Proofs are left to the reader)