Some examples

RECOGNITION PRINCIPLE. Let $H \subseteq GL(n, \mathbb{R})$ be a (closed) subgroup which is the zero set of a smooth function $\varphi : GL(n, \mathbb{R}) \longrightarrow \mathbb{R}^p$, and assume that $D\varphi(I)$ is onto. Then H is a Lie subgroup; i.e., H is a smooth submanifold of $GL(n, \mathbb{R})$.

Proof. By the Inverse Function Theorem, there is a neighborhood W of I such that $H \cap W$ is a smooth submanifold. By symmetry, if $h \in H$, then $H \cap h \cdot W = hH \cap h \cdot W$ is a smooth submanifold of $h \cdot W$. In fact, if $\varphi_h = \varphi \circ L_{h^{-1}}$ then H is the zero set of φ_h and $D\varphi_h(h)$ is onto. We can then conclude that H is a smooth submanifold by the following generalization of the result on level sets:

THEOREM. Suppose that $N \subseteq \mathbb{R}^q$ is a closed subset with the the following property:

(*) For each $x \in N$ there is a smooth chart $k: W \to U \times V$

where U is open in $\mathbb{R}^n$, V is an open neighborhood of 0 in $\mathbb{R}^{q-n}$ such that $N \cap W$ corresponds to $U \times \{0\}$.

Then N is a smooth submanifold of $\mathbb{R}^q$.

Basic examples

Checking that $D\varphi(I)$ is onto will be left to the reader in some cases.

1. $SL(n, \mathbb{R}) \subseteq GL(n, \mathbb{R})$ the set of matrices with determinant 1.

In this case the hypothesis on $\varphi = \det$ can be checked as follows: Let $\gamma: (-h, h) \to GL(n, \mathbb{R})$ be given by

$$\gamma(t) = \begin{array}{c c} & \begin{array}{cc} 1 & n-1 \end{array} \\ \begin{array}{c} 1 \\ n-1 \end{array} & \left(\begin{array}{c|c} e^t & 0 \\ \hline 0 & I \end{array}\right) \end{array}$$

Then $\frac{d}{dt} \det(\gamma(t)) \neq 0$.

The latter implies that $D\det(I)$ has rank 1.

orthogonal matrices

2. $O(n) \subseteq GL(n, \mathbb{R})$ Let $\text{Sym}(n) \subseteq$ $n \times n$ matrices be the set of all $n \times n$ matrices which are symmetric, so $\dim \text{Sym}(n) = \binom{n+1}{2}$ and $\text{Sym}(n) \cong \mathbb{R}^p$ for $p =$ this value. Take T to be $({}^tA)A - I$.

transpose

For example 1, note that the tangent space to the identity is all matrices with trace 0,* and for example 2, the tangent space to the identity is the set of skew-symmetric matrices.

[* This might be hard to check directly.]

3. $UT(n) \subseteq GL(n, \mathbb{R})$, the set of upper unitriangular matrices A with $a_{ii} = 1$ (all i) $a_{ij} = 0$ if $i < j$. One can check directly that $UT(n)$ is a subgroup and is diffeomorphic to $\mathbb{R}^p$ where $p = \binom{n-1}{2}$.

4. $GL(n, \mathbb{C}) \subseteq GL(2n, \mathbb{R})$.

Suppose that $\mathbb{R}^{2n}$ has coordinates $(x_1, x_2, \ldots, x_{2n-1}, x_{2n})$. The defining equations are $a_{2k-1,2k-1} = a_{2k,2k}$ and $a_{2k-1,2k} = -a_{2k,2k-1}$. Recall that we can view a complex number $a+bi$ as a 2×2 matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

5. $U(n) \subseteq GL(n, \mathbb{C}) \subseteq GL(n, \mathbb{R})$.

Use the preceding conditions plus the defining equation for $U(n)$: $A^*A = I$, where A^* is the conjugate transpose of A.

6. $SU(n) \subseteq U(n)$. SU_n = all matrices with determinant 1.

Add the condition $\det A = 1$ to the conditions in 4 and 5.

Note that $O(n)$, $U(n)$ and $SU(n)$ are compact Lie groups. Also, $U(1) = S^1$, $U(1) =$ identity component of $O(2)$.

Addendum

Finding the tangent space to $SL(n,\mathbb{R}) \subseteq GL(n,\mathbb{R})$ at the identity.

We claim the tangent space is the set of all $n \times n$ matrices with trace $= 0$.

The first step is the following identity.

FORMULA. $\det(\exp(A)) = e^{\text{trace}(A)}$.

Sketch of proof. Since $\exp(P^{-1}AP) = P^{-1}\exp(A)P$ and both det and tr are equal for similar matrices, it suffices to prove this for a class of matrices which contains examples from every similarity class. It also suffices to work over the complex numbers, and by the preceding considerations it suffices to prove the result for matrices in Jordan form. For such matrices one can check this directly. ∎

PROPOSITION. The map $A \to \exp(A)$, on $n \times n$ matrices, maps a neighborhood of 0 diffeomorphically to a neighborhood of I.

Sketch of proof. One can use the power series for $\exp(A)$ to prove directly that $D\exp(I)$ is the identity. ■

PROOF OF THE ASSERTION ON TANGENT SPACES

If V is the tangent space to $SL(n, \mathbb{R})$ at I, then $\dim V = n^2 - 1$. Suppose now that W is the kernel of the trace homomorphism, so that $\dim W$ also equals n^2-1. It suffices to show that $W \subseteq V$.

Given $A \in W$, let $\gamma(t) = \exp(tA)$. By the formula on the preceding page, trace $A = 0$ $\Rightarrow \gamma(t) \in SL(n, \mathbb{R})$. Since $\gamma'(0) = A$, it follows that $A \in V$; therefore $W \subseteq V$, as required. ■