8. Vector fields

Chapters 8 and 9 of Lee discuss the basic facts about (tangent) vector fields. However, there is only enough time in the course to cover the first of these chapters.

Let $U \subseteq \mathbb{R}^n$ be open. In multivariable calculus a vector field is defined to be a smooth map $F: U \longrightarrow \mathbb{R}^n$.

There are two contexts in which vector fields arise in the basic theory of smooth manifolds:

① Generalizations of directional derivatives, where the direction varies from point to point. (Chapter 8)

② Velocity fields which describe the flow of a fluid in which the velocity vectors are constant over time. (Chapter 9). Formally, the flow curves are solutions to the system of ordinary differential equation

$$\frac{dx}{dt} = F(x).$$

We can only look at ① here, but first we shall give some basic definitions & facts.

<u>**Def.**</u> Let M^n be a smooth manifold. A <u>vector field</u> X on M^n is a mapping $X: M^n \longrightarrow T(M^n)$ such that $\tau_M \circ X = id_M$.

Generally X will be at least continuous, but usually we assume X is smooth.

In words, a vector field assigns a tangent vector at p to each point p of a smooth manifold.

<u>Local coordinates</u> If $M^n = U$ is open in $\mathbb{R}^n$, a vector field is given by $(x, F(x)) \in U \times \mathbb{R}^n$ for some $F: U \longrightarrow \mathbb{R}^n$.

<u>Example</u> For every M^n, we have the zero vector field which is smooth and sends $p \in M^n$ to the zero vector $0_p \in T_p(M^n)$.

Proposition. Pointwise addition and "scalar" multiplication make the set $VF(M^n)$ of vector fields into a module over the ring of smooth functions $C^\infty(M)$.

Note that if U is open in M, then there is a natural restriction map $VF(M^n) \longrightarrow VF(U)$. $X \to T(U)|X|U$

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Extension Principle. Suppose that A is closed in M, U is an open set in M containing A, and $X_0 \in VF(U)$. Then there is an open nbhd. W of A such that $A \subseteq \Omega \subseteq \overline{\Omega} \subseteq U$ and a vector field X on M such that "$X|\Omega$" = "$X_0|\Omega$."

In particular, there are many examples of nonzero vector fields.

Idea of proof. Choose open sets so that
$A \subseteq \Omega \subseteq \overline{\Omega} \subseteq W \subseteq \overline{W} \subseteq U$, and let
$\varphi: U \to \mathbb{R}$ be a smooth function which is

1 on $\overline{\Omega}$ and 0 on $W\,U-\overline{W}$. Then $\varphi \cdot X_0$ extends to a vector field on M which is zero on $M-\overline{W}$. By construction, the restrictions of X and X_0 to Ω are equal. ∎

<u>NOTATION.</u> $U \subseteq \mathbb{R}^n$ open, $e_i \in \mathbb{R}^n$ is the standard unit vector. Then the vector field $\frac{\partial}{\partial x_i}$ is the map sending $p \longrightarrow (p, e_i)$.

The reason for this notation will be explained shortly.

<u>CONSEQUENCE.</u> Every smooth $X \in VF(U)$ is uniquely expressible as a "linear combination"

$$\sum f_i \frac{\partial}{\partial x_i} \quad \text{for some } f_i \in C^\infty(U).$$

(And similarly for continuous vector fields)

Clearly there are many smooth vector fields on U which are nowhere zero.

On the other hand, we have

THEOREM. If $X \in VF(S^{2n})$, then there is some $p \in S^{2n}$ such that $X(p) = 0$.

(See Hatcher, Thm. 2.28, p. 135.)

Def. M^n is *parallelizable* if there are vector fields $X_1, \ldots, X_n$ on M^n such that for each $p \in M^n$ the vectors $\{X_i(p)\}$ form a basis for $T_p(M)$.

A fairly deep result of Bott and Milnor shows that S^n is parallelizable $\iff n = 1, 3, 7$.