Vector fields and directional derivatives

Def. M^n smooth manifold, $X \in VF(M^n)$ (smooth) and $f \in C^\infty(M)$. The $\left\{\begin{array}{c}\text{directional}\\ \text{Lie}\end{array}\right\}$ derivative of f along X is the composite

$$Xf : M \xrightarrow{X} T(M) \xrightarrow{T(f)} T(\mathbb{R}) \cong \mathbb{R}\times\mathbb{R} \xrightarrow{\text{PROJ}_2} \mathbb{R}.$$

(Note: PROJ_1 corresponds to $\tau_{\mathbb{R}} : T(\mathbb{R}) \to \mathbb{R}$.)

LOCAL FORMULA. If $M = U$ is open in $\mathbb{R}^n$, then $X = \sum f_i \frac{\partial}{\partial x_i}$ for suitable $f_i \in C^\infty(U)$, and

$$Xg = \sum f_i \frac{\partial g}{\partial x_i}.$$

(and this is the reason for the $\frac{\partial}{\partial x_i}$ notation!)

PROPOSITION. The map $g \longrightarrow Xg$ is a derivation:

$$X(g_1+g_2) = Xg_1 + Xg_2$$
$$X(cg) = c\,Xg \quad (c \text{ constant})$$
$$X(g_1 g_2) = (Xg_1)\,g_2 + g_1\,(Xg_2).$$

(One can prove this by looking locally.)

Generalities on derivations

Def. Let A be a vector-space over a field $\mathbb{F}$ in which $1+1 \neq 0$. An algebra structure on A is a multiplication $A \times A \longrightarrow A$ such that $(a_1+a_2)b = a_1 b + a_2 b$, $a(b_1+b_2) = ab_1 + ab_2$, and $(ca)b = c(ab) = a(cb)$ [where $a_i, b_i, a, b \in A$ and $c \in \mathbb{F}$.]

① A is associative if $(a_1 a_2)a_3 = a_1(a_2 a_3)$ all a_i.

② A is Lie if $ab = -ba$ (equivalently, $a^2 = 0$ *)

Jacobi Identity → and $a(bc) + (ab)c + c(ab) + b(ca) = 0$.

$\boxed{\text{* this is true because } 1+1 \neq 0 \text{ in } \mathbb{F}}$

(other types of algebras arise in mathematics, but we need not consider them here).

EXAMPLES. 1. If all products are 0, we have an abelian Lie algebra (this is turns out to be a very useful concept!).

2. $\mathbb{R}^3$ with the cross product. — The Jacobi Identity follows from three applications of the "BAC-CAB" Rule:

$$a \times (b \times c) = b(a \cdot c) - c(a \cdot b).$$

(permute a, b, c cyclically)

3. If A is an associative algebra, then one obtains a Lie algebra $\mathcal{L}(A)$ with the same vector space structure and

Commutator $\longrightarrow$ $[a, b] = ab - ba$. [Details left to the reader]

The following result is related to Example 3.

Def. Let A be an associative algebra over $\mathbb{F}$. A linear transformation $D : A \longrightarrow A$ is a *derivation* if $D(a_1 a_2) = a_1 (D a_2) + (D a_1) a_2$. Leibniz Rule

THEOREM. Let A be as above, and let $\mathcal{E}(A)$ be the associative algebra of linear transformations $A \to A$. If $\mathcal{D}(A) \subseteq \mathcal{E}(A)$ is the set of derivations, then $\mathcal{L}\mathcal{D}(A)$ is a Lie subalgebra of $\mathcal{L}\mathcal{E}(A)$.

(Verification is left to the reader.)

Vector fields and systems of derivations.

Let M^n be a smooth manifold. A system of derivations on M^n is a family of derivations $D_U \in \mathcal{D}(C^\infty(U))$, where U runs through the open subsets of M, such that if $i : U \subseteq V$ is an inclusion then

$$D_U(f|U) = (D_V f)|U.$$ compatibility identity

PROPOSITION. Every smooth vector field on M defines a system of derivations, and if X, X' define the same derivation on M, then $X = X'$ (and likewise over each $U \subseteq M$).

<u>Sketch of proof.</u> The system of derivations is given by taking Lie derivatives along the vector field. This map is $\mathbb{R}$-linear, so it suffices to consider the case where $X' = 0$. By the compatibility identity the latter reduces to checking the case where U is open in $\mathbb{R}^n$. In this situation we may write $X = \sum w_i \frac{\partial}{\partial x_i}$ for suitable functions w_i. Now if g_j is the jth coordinate we have $X g_j = w_j$, and if X induces the zero derivation we must have $w_j = 0$. Letting j run from 1 to n, we see that $X = 0$. ∎

<u>THEOREM.</u> Every system of derivations is defined by a smooth vector field.

<u>Proof</u> <u>CLAIM</u>: It will be enough to prove this for an open disk U in $\mathbb{R}^n$. If this case is known, take a nice open covering of M^n by

open sets U_α diffeomorphic to disks. This yields locally defined vector fields X_α on the sets U_α. Since two vector fields are equal if they determine the same derivation, it follows that $X_\alpha | U_\alpha \cap U_\beta = X_\beta | U_\alpha \cap U_\beta$ for all α, β. Therefore the X_α can be assembled into a global vector field X on M, and it follows that D is determined by X.

Solving the local problem.

Recall that we are now assuming that U is an open disk in $\mathbb{R}^n$.

Let $y_j : U \to \mathbb{R}$ be the jth coordinate function, and let $g_j = D y_j$. CLAIM: D is given by the vectorfield $\sum g_j \frac{\partial}{\partial x_j} = X$.

Let $a \in U$, $a = (a_1, \ldots, a_n)$, and choose an open disk V such that a is its center and $V \subseteq U$. Then the exact remainder (integral form) of Taylor's Theorem implies that we have

$$f(y) = f(a) + \sum \frac{\partial f}{\partial y_i}(a) \cdot [y_i - a_i] +$$

$$\sum_{i,j} [y_i - a_i] \cdot [y_j - a_j] \, S_{ij}(y)$$ ← smooth fcn.

for $y \in V$. Let $\Delta y_j = y_j - a_j$

Lemma Suppose D is a derivation on $C^\infty(W)$. $a \in W$

① $D(\text{constant}) = 0$

② If $u(a) = v(a) = 0$, then $D(uv)(a) = 0$.

(See Lee, p. 54, for a proof).

Returning to the local problem, we then have

$$Df(y) = D\Big(\underbrace{f(a) + \sum \frac{\partial f}{\partial y_i}(a)}_{\text{constants}} \Delta y_i + \sum \Delta y_i \Delta y_j \, S_{ij}(y) \Big)$$

$$= \sum \frac{\partial f}{\partial y_i}(a) \underbrace{D \Delta y_i}_{g_i} + \sum \underbrace{(\Delta y_i \Delta y_j)}_{\text{zero if } y = a} DS_{ij} + \sum \underbrace{D(\Delta x_i \Delta y_j)}_{\text{also zero at } a \text{ by } ②} \cdot S_{ij}$$

If we now set $y = a$ we see that

$$Df(a) = \sum g_i(a) \frac{\partial f}{\partial y_i}(a) = Xf(a)$$

which is what we wanted to prove. ∎