Lie brackets of vector fields

The preceding results lead to the following conclusion.

THEOREM. Let X and Y be smooth vector fields on M. Then there exists a vector field $[X, Y]$ which is uniquely determined by the property $[X, Y]f = X(Yf) - Y(Xf)$ for all $f \in C^\infty(M)$. This operation yields a Lie algebra structure on the real vector space $VF(M)$.

Proof. Since the commutator of a derivation is also a derivation, the construction

$$Df = X(Yf) - Y(Xf)$$

induces a system of derivations over M (one must check the compatibility condition, but this is straightforward). Therefore D is determined by some $Z \stackrel{\text{DEF}}{=} [X, Y]$ on M. ∎

Properties of this $\boxed{\text{Lie bracket}}$ operation.

1. $[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y]$.

2. $[X, Y] = -[Y, X]$.

3. If u and v are smooth functions, then

$$[uX, vY] = uv[X, Y] + \underbrace{(uXv)Y - (vYu)X}.$$

$\left(\text{but it is } \mathbb{R}\text{-linear}\right)$ $\longleftarrow$ So $[\,,\,]$ is not $C^\infty(M)$ bilinear!!

All of these can be checked directly by looking at the associated derivations.

Computing Lie brackets in coordinates

Suppose U is open in $\mathbb{R}^n$ and the vector fields are given in coordinates by

$$X = \sum u_i \frac{\partial}{\partial x_i} \qquad Y = \sum v_j \frac{\partial}{\partial x_j}$$

Then one can compute $[X, Y]$ as follows:

First, expand the bracket into a sum

$$\sum_{i,j} \left[u_i \frac{\partial}{\partial x_i}, v_j \frac{\partial}{\partial x_j} \right]$$ by distributivity.

Next, evaluate each $\left[u_i \frac{\partial}{\partial x_i}, v_j \frac{\partial}{\partial x_j} \right]$ using property ③ above. $\left(u_i \frac{\partial v_j}{\partial x_i}\right)\frac{\partial}{\partial x_j} - \left(v_j \frac{\partial u_i}{\partial x_j}\right)\frac{\partial}{\partial x_i}$.

Notice that all terms $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right]$ vanish.

This is true by anticommutativity and $0 \neq 1+1 \in \mathbb{R}$ if $i=j$, and if $i \neq j$ this follows from equality of mixed partial derivatives (for ∞ differentiable functions)

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Finally, simplify the expression by combining like terms whenever possible.

Vector fields and diffeomorphisms

We have tacitly used the fact that if M and N are diffeomorphic, then $VF(M) \cong VF(N)$ as $\mathbb{R}$-vector spaces. At this point we need to formulate this fact more precisely.

smooth vector fields

Def. Let $\varphi : M \longrightarrow N$ be a diffeomorphism. The map $\varphi_* : VF(M) \longrightarrow VF(N)$ is defined by sending a vector field X on M to $\varphi_* X$:

$$N \xrightarrow{\varphi^{-1}} M \xrightarrow{X} T(M) \longrightarrow T(N) \ (= \varphi_* X),$$

so that we have a commutative diagram

$$\begin{tikzcd} T(M) \arrow[r, "T(\varphi)", "\cong"'] & T(N) \\ M \arrow[u, "X"] \arrow[r, "\varphi", "\cong"'] & N \arrow[u, "\varphi_* X"'] \end{tikzcd}$$

This is the unique vectorfield which makes the digram commute.

It is elementary to check the following:

① $id_* = \text{identity}$, $(\varphi_*)^{-1} = (\varphi^{-1})_*$ $(\varphi_1 \varphi_2)_* = \varphi_{1*} \varphi_{2*}$

② φ_* is a linear transformation of $\mathbb{R}$ vector spaces.

It is also worthwhile to understand how φ_* behaves on Lie derivatives:

FORMULA. Let $f \in C^\infty(N)$ and $X \in VF(M)$. If $\varphi : M \longrightarrow N$ is a diffeomorphism, then

$$\left[\varphi_* X\right](f) = \left(\left[X\right](f\varphi)\right) \circ \varphi^{-1}.$$

Proof By definition, the left side is the composite of the top line in the commutative diagram below:

$$\begin{tikzcd}
N \arrow[r, "\varphi_* X"] & T(N) \arrow[r, "T(f)"] & T(\mathbb{R}) \cong \mathbb{R}\times\mathbb{R} \arrow[r, "\mathrm{PROJ}_2"] & \mathbb{R} \\
M \arrow[u, "\varphi", "\cong"'] \arrow[r, "X"] & N \arrow[u, "T(\varphi)", "\cong"'] \arrow[ur, "T(f\circ\varphi)"'] & &
\end{tikzcd}$$

By the definition of $\varphi_* X$ we have a commutative square at the left, and since $T(f\circ\varphi) = T(f)\circ T(\varphi)$ the formula follows from the commutativity of the entire displayed diagram. ∎

THEOREM. If $\varphi: M \to N$ is a diffeomorphism then $\varphi_*[X,Y] = [\varphi_* X, \varphi_* Y]$ for all $X, Y \in VF(M)$. In words, φ_* is an isomorphism of Lie algebras.

Proof. It suffices to check that for all $f \in C^\infty(N)$ we have $[\varphi_* X, \varphi_* Y] f = (\varphi_*[X,Y]) f$ and since φ is a diffeomorphism this is equivalent to the identity $([\varphi_* X, \varphi_* Y] f) \circ \varphi = \Big((\varphi_*[X,Y]) f\Big) \circ \varphi$. We shall use the following equivalent version of the Formula on the preceding page: $((\varphi_* Z) f) \circ \varphi = Z(f\varphi)$ for all $Z \in VF(M)$.

Proceeding with the verification, we have

$$((\varphi_*[X,Y]) f) \circ \varphi = [X,Y](f\varphi) =$$
$$X(Y(f\varphi)) - Y(X(f\varphi)) =$$
$$X(((\varphi_* Y) f) \circ \varphi) - Y(((\varphi_* X) f) \circ \varphi) =$$
$$(\varphi_* X((\varphi_* Y) f)) \circ \varphi - (\varphi_* Y((\varphi_* X) f)) \circ \varphi =$$

$$\left(\varphi_* X((\varphi_* Y)f) - \varphi_* Y((\varphi_* X)f)\right) \circ \varphi =$$
$$\left([\varphi_* X, \varphi_* Y] f\right) \circ \varphi$$

and as noted before this verifies the Lie bracket identity. ∎