Applications to Lie groups

Def. If G is a Lie group, a vector field X on M is said to be $\boxed{\text{left invariant}}$ if $L_{a*}X = X$ for all $a \in G$.

By the preceding results, the set of left-invariant vector fields on G is a Lie subalgebra of $VF(G)$.

FINITE DIMENSIONALITY. A left invariant vector field is completely determined by its value at $1 \in G$.

Proof. Left invariance implies the following diagram is commutative for every $g \in G$

$$\begin{tikzcd} T(M) \arrow[r, "T(L_g)"] & T(M) \\ M \arrow[u, "X"] \arrow[r, "L_g"] & M \arrow[u, "X"'] \end{tikzcd}$$

Therefore $X(g) = X L_g(1) = T(L_g)X(1)$. ∎

CONVERSE. If $A \in \mathfrak{g} = T_1(G)$, then there is a (unique) left invariant vector field X^A with $X^A(1) = A$. Furthermore the map $A \longrightarrow X^A$ is $\mathbb{R}$-linear.

SKETCH OF PROOF. Let $A \in \mathfrak{g}$, and define X^A by

$$\begin{tikzcd}
G \arrow[r, "\cong"] \arrow[drrr, bend right, "{X^A,\ \text{smooth by construction}}"'] & G \times \{A\} \arrow[r] & T(G) \times T(G) \arrow[r, "\text{NAT}", "\text{ISO}"'] & T(G \times G) \arrow[d, "T(\text{mult.})"] \\
 & & & T(G).
\end{tikzcd}$$

$$(g, A) \longrightarrow (0_g, A)$$

zero vector

Then one can check the following:

$$\tau_G \circ X^A = id_G$$

$$X^A(1) = A$$

$$g_* X^A = X^A \quad \text{all } g \in G$$

$$X^A(g) = T(L_g)(A)$$

$$X^{cA+B} = cX^A + X^B$$

Taken together, they show that X^A has all the required properties. ∎

<u>Def.</u> If $A, B \in \mathfrak{g}$, then $[A, B] \in \mathfrak{g}$ is the unique C such that the left invariant vector field $[X^A, X^B]$ is equal to X^C.

It follows that we have a Lie algebra structure on $\mathfrak{g}$.

<u>Example</u> If $G = GL(n, \mathbb{R})$, so that $\mathfrak{g} \longleftrightarrow$ the matrix algebra $M_n(\mathbb{R})$, then this Lie structure is given by the matrix commutator product $[A, B] = AB - BA$.

(See Lee, pp. 193-195, for a proof.)

The Lie bracket on $\mathfrak{g}$ carries an enormous amount of information about G, and this is discussed further in Chapters 8 and 20 of Lee. In particular, Thm. 20.21 in Lee describes a <u>1-1</u> correspondence between <u>simply connected</u> Lie groups and Lie algebras over $\mathbb{R}$.