EXERCISES FOR MATHEMATICS 205C

SPRING 2005

The references denote sections of the texts for this course and the two preceding courses in the sequence:

L. Conlon, *Differentiable Manifolds* (Second Edition), Birkhäuser-Boston, Boston MA, 2001, ISBN 0–8176–4134–3.

J. R. Munkres, *Topology* (Second Edition), Prentice-Hall, Saddle River NJ, 2000, ISBN 0–13–181629–2.

I. Topological Background

I.1 : Topological manifolds

(Conlon, §§ 1.1–1.2, 1.7, Appendix A)

Additional exercise

1. Let X be a topological space. Prove that X is a topological 0-manifold if and only if X is discrete.

2. Suppose that X is a topological manifold and U is an open subset of X. Prove that U is a topological manifold.

3. Let X be a Hausdorff space, and suppose that X has an open covering $\{U_\alpha\}$ such that each U_α is a topological n-manifold for some fixed n. Prove that X is a topological manifold. Give a counterexample to this statement if the Hausdorff condition is removed.

4. (*i*) Suppose that X is a topological n-manifold and Y is a topological m-manifold. Prove that $X \times Y$ with the product topology is a topological $(m+n)$-manifold.

(*ii*) Suppose that E and X are connected Hausdorff spaces and topological n-manifold if and only if X is.

5. Let X be a Hausdorff space, and suppose that X has an open covering $\{U_\alpha\}$ such that each U_α is a topological n-manifold for some fixed n. Prove that X is a topological manifold. Why is this false if the Hausdorff condition is removed?

6. A compact Hausdorff space Γ is a *graph* if it is a finite union of subspaces E_j such that each E_j is homeomorphic to the closed unit interval $[0,1]$ and if $i \neq j$ then $E_i \cap E_j$ is an endpoint of both E_i and E_j (note that one can characterize the endpoints topologically as the two points whose complements are connected). The set of endpoints of the subsets E_k is called the set of vertices of Γ and each E_k is called an edge of Γ. Prove that if Γ is a topological manifold, then every vertex lies on exactly two edges. [*Hint:* Look at the proof that the figure 8 curve is not a topological manifold.]

7. In the notation of the previous exercise, it follows that if one removes a finite set of points from Γ, then Γ is a topological 1-manifold. All letters in the alphabet and all Hindu-Arabic numerals admit decompositions into closed subspaces which make them into graphs. Assuming that the letters and numerals are given in the sans-serif form

A B C D E F G H I J K L M

N O P Q R S T U V W X Y Z

1 2 3 4 5 6 7 8 9 0

determine the least numbers of points that must be removed in order to obtain a topological manifold.

I.2 : Partitions of unity

(Conlon, §§ 1.4–1-5)

Additional exercises

1. (*i*) For each positive integer n let V_n be the open annulus (ring-shaped region) consisting of all points x such that $n-2 < |x| < n+1$. Prove that the family of subsets $\{ V_n \}$ is locally finite.

(*ii*) Suppose X is a topological n-manifold and $\mathcal{U}$ is an open covering of X. Prove that there is an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that every subset of $\mathcal{V}$ is homeomorphic to $\mathbf{R}^n$.

2. Suppose that U is open in $\mathbf{R}^n$ and W is an open neighborhood of $U \times \{0\}$ in $U \times [0,1)$. Prove that there is an open subneighborhood $W_0 \subset W$ of U such that U is a deformation retract of W_0. [*Hint:* Partitions of unity guarantee the existence of a continuous positive valued function f on U whose graph is completely contained in W. Consider the open set W_0 of all points in $U \times [0,1)$ that lie under the graph of f.]

3. Let M be a second countable topological manifold and let $M^\bullet$ denote its one point compactification. Using the metrization theorems for topological manifolds, prove that $M^\bullet$ is also second countable and metrizable. [*Hint:* First prove that $A^\bullet$ is second countable because it is σ-compactness. For metrizability, suppose more generally that A is a bounded locally compact subset of the normed vector space $\mathbf{R}^\infty$ and $\varphi : A \to [0, +\infty) \cong [0,1)$ is a proper map. Consider the function $f : A^\bullet \to \mathbf{R}^\infty \times [0,1]$ defined by

$$f(z,t) \quad = \quad \big([1 - \varphi(z)] \cdot f(z),\, 1 - \varphi(z) \big)$$

for ordinary points $z \in A$ and $f(\infty_A) = (0,1)$. Prove that this map is continuous on all of A; there are two cases depending upon whether one has an ordinary point or ∞_A. Also verify that f is 1–1. Why do these properties suffice to show that f maps $A^\bullet$ homeomorphicall onto its image?]

4. [*This question requires some background knowledge from measure theory.*] Given a second countable topological manifold X, define the family of *Borel sets* $\mathcal{B}$ in X to be the smallest family of subsets that contains the open subsets and is closed under the operations of countable union, countable intersection and complementation (so it follows that $\mathcal{B}$ also contains all closed subsets). We shall say that a nonnegative measure on $\mathcal{B}$ is *topologically well-behaved* if (*i*) all one point subsets have measure zero, (*ii*) all open subsets have positive measure. — For each $n > 0$, the standard Lebesgue measure on $\mathbf{R}^n$ defines a topologically well-behaved (Borel) measure.

(i) Show that $\mathcal{B}$ consists of countable unions of countable intersections $\bigcap_n D_n$ where each D_n is either open or closed in X. [*Hint:* Verify that the family of such subsets is closed under countable unions and intersections as well as complementation.]

(ii) If X is a topological n-manifold for some $n > 0$, prove that X has a topologically well-behaved Borel measure. [*Hint:* If U is an open subset of X and there is a homeomorphism $h : V \to U$ where V is open in $\mathbf{R}^n$, why does h send the Borel subsets of V to the Borel subsets of U and vice versa? Show that one can define a measure m_U on U by setting $m_U(A) = |h^{-1}(A)|$ where $|\cdots|$ denotes the usual Lebesgue measure on $\mathbf{R}^n$. Why is this a topologically well-behaved measure? Finally, show that if one pieces a suitable collection of such local measures together using a partition of unity then one obtains a Borel measure with the desired properties.]

REMARK. The so-called Oxtoby-Ulam Theorem gives an interesting characterization of the standard Lebesgue measure on $\mathbf{R}^n$: If λ is a Borel measure on $\mathbf{R}^n$ such that (i) all one point subsets have measure zero, (ii) all nonempty open subsets have positive measure, then there is a homeomorphism h from $\mathbf{R}^n$ to itself such that $\lambda(A) = |\,h^{-1}(A)\,|$, where $|\cdots|$ denotes the usual Lebesgue measure on $\mathbf{R}^n$. Here are some references:

[1] J. C. Oxtoby and S. M. Ulam, *measure preserving homeomorphisms and metrical transitivity*, Ann. of Math. (2) **42** (1941), 874–920.

[2] C. Goffman, *A note on integration*, Math. Mag. **44** (1971), 1–4.

[3] C. Goffman, T. Nishiura, and D. Waterman, Homeomorphisms in Analysis, Mathematical Surveys and Mongraphs No. 54. American Mathematical Society, Providence RI, 1997. ISBN: 0-8218-3214-X. — Also available online from the following site:

`http://www.ams.org/online_bks/surv54/`

I.3 : The Contraction Lemma

(Conlon, Appendix B)

Additional exercises

1. Prove that the equation $2 - x - \sin x = 0$ has a real root and that it lies in the closed interval with endpoints $\pi/6$ and $\pi/2$. Show that $\varphi(x) = 2 - \sin x$ is a contraction operator on this interval and then find the root, accurate to six decimal places. [In this example it might be worthwhile to use a scientific calculator to estimate the numerical value.]

2. Let X be a metric space. A map $f : X \to X$ is said to be a *nonisometric* (or proper) *similarity* of X if f is onto and there is a positive constant $C \neq 1$ such that

$$\mathbf{d}\,(\,f(u),\,f(v)\,) \quad = \quad C \cdot \mathbf{d}(u,v)$$

for all u, $v \in X$ (hence f is 1–1 and uniformly continuous, and in fact has a uniformly continuous inverse that is also a proper similarity). Prove that every nonisometric similarity of a complete metric space has a unique fixed point. [*Hint and comment:* Split into two cases depending upon whether $C < 1$ and $C > 1$. In the first case the surjectivity condition turns out to be unnecessary. In the second case, verify that f has an inverse that is uniformly continuous. Why does $f(x) = x$ hold if and only if $f^{-1}(x) = x$? — The most elementary examples of such maps arise when $X = \mathbf{R}^n$ and a classical geometric similarity is given by $f(x) = cAx + b$, where A comes from an orthogonal matrix and either $0 < C < 1$ or $C > 1$; for these examples one can prove the existence of a unique fixed point using elementary linear algebra.]

I.4 : Basic topological constructions revisited

(*cf.* [MUNKRES1], §§ 15, 16, 19, 22)

Additional exercises

1. (*i*) Let X, Y and Z be sets (*resp.*, topological spaces), and let $\times$ denote the usual cartesian product. Prove that

$$X \times (Y \times Z)$$

is a direct product of sets (*resp.*, topological spaces) as defined in the notes.

(*ii*) Let A, B, C and D be sets (*resp.* topological spaces), and let $\times$ denote the usual cartesian product. Prove that

$$(A \times B) \ \times \ (C \times D)$$

is a direct product of sets (*resp.* topological spaces) as defined in the notes.

[*Note:* These may all be viewed as special cases of a more general result.]

2. (*i*) Let X and Y be topological spaces and let $\tau : X \times Y \to Y \times X$ be the "twist map" which sends (x, y) to (y, x) for all x and y. Prove that τ is a homeomorphism. [*Hint:* Consider the analogous map $\tau' : Y \times X \to X \times Y$.]

(*ii*) Let X be a topological space and let $T : X \times X \times X \to X \times X \times X$ be the map that cyclically permutes the coordinates: $T(x, y, z) = (z, x, y)$. Prove that T is a homeomorphism. [*Hint:* What is the test for continuity of a map into a product? Can you write down an explicit formula for the inverse function?]

3. ("A product of products is a product.") Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of nonempty sets, and let $\mathcal{A} = \cup\{\mathcal{A}_\beta \mid \beta \in \mathcal{B}\}$ be a partition of $\mathcal{A}$. Construct a bijective map of $\prod\{A_\alpha \mid \alpha \in \mathcal{A}\}$ to the set

$$\prod_\beta \{\prod\{A_\alpha \mid \alpha \in \mathcal{A}_\beta\} \ \} \ .$$

If each A_α is a topological space and we are working with product topologies, prove that this bijection is a homeomorphism.

4. Let A be some nonempty set, let $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ and $\{Y_\alpha \mid \alpha \in \mathcal{A}\}$ be families of topological spaces, and for each $\alpha \in A$ suppose that $f_\alpha : X_\alpha \to Y_\alpha$ is a homeomorphism. Prove that the product map

$$\prod_\alpha f_\alpha \ : \ \prod_\alpha X_\alpha \longrightarrow \prod_\alpha Y_\alpha$$

is also a homeomorphism. [*Hint:* What happens when you take the product of the inverse maps?]

5. (*i*) Suppose that $f : X \to Y$ and $g : Y \to Z$ are topological embeddings. Prove that $g \circ f$ is also a topological embedding.

(*ii*) Suppose that $h : A \to X$ and $k : B \to Y$ are topological embeddings. Prove that $h \times k : A \times B \to X \times Y$ is also a topological embedding.

II. Local theory of smooth functions

II.1 : Differentiability

(Conlon, §§2,1, 2.3–2.4)

Additional exercises

1. Use the derivative approximation to estimate the following:

(*i*) $[(3.02)^2 + (1.97)^2 + (5.98)^2]$

(*ii*) $(e^4)^{1/10} = \exp\left((1.1)^2 - (0.9)^2 \right)$

2. Let $f : \mathbf{R}^n \to \mathbf{R}$ be differentiable. If $f(0) = 0$ and $f(tx) = tf(x)$ for all t and x prove that $f(x) = \langle \nabla f(0), x \rangle$ for all x; *i.e.*, f is linear. Consequently, any nonlinear function g satisfying the conditions $g(0) = 0$ and $g(tx) = t\,g(x)$ for all t and x is not differentiable although it has directional derivatives in all directions at the origin (why?).

3. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x,y) \quad = \quad \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

(*i*) Show that $D_1 f(0,y) = -y$ and $D_2 f(x,0) = x$ for all x and y.

(*ii*) Conclude that $D_1 D_2 f(0,0)$ and $D_2 D_1 f(0,0)$ exist but are not equal.

4. Show that each of the following is a solution of the *heat equation*

$$\frac{\partial u}{\partial t} \quad = \quad k^2 \frac{\partial^2 u}{\partial x^2}$$

(where k is a constant):

(*i*) $\exp(-k^2 a^2 t) \sin ax$

(*ii*) $\exp(-x^2/4k^2 t)/\sqrt{t}$

5. (*i*) If $f(x) = g(\rho)$ where $\rho = |x|$ and the number n of variables is at least 3, show that

$$\nabla^2 f \quad = \quad \frac{n-1}{\rho}\, g'(\rho) + g''(\rho)$$

for $x \neq 0$.

(b) Using the formula displayed above, prove that if $\nabla^2 f = 0$ then

$$f(x) \quad = \quad \frac{a}{|x|^{n-2}} + b$$

where $x \neq 0$ and a and b are constants.

6. Verify that the functions $r^n \cos^n \theta$ and $r^n \sin^n \theta$ satisfy the 2-dimensional Laplace equation in polar coordinates. [Exercise 3.9 on the same page gives the formula for the Laplacian in polar coordinates.

7. If

$$f(x, y, z) \quad = \quad \frac{1}{\rho} \cdot g\left(t - \frac{\rho}{c}\right)$$

where $\rho = (x^2 + y^2 + x^2)^{1/2}$ and c is a constant, show that f satisfies the 3-dimensional *wave equation*

$$\nabla^2 f \quad = \quad \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} \, .$$

8. The following shows the hazards of denoting functions by real variables. Let $w = f(x, y, z)$ and $z = g(x, y)$. Then

$$\frac{\partial w}{\partial x} \quad = \quad \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} \quad = \quad \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}$$

because the partials of x and y with respect to x are 1 and 0 respectively. Therefore

$$\frac{\partial w}{\partial z}\frac{\partial z}{\partial x} = 0$$

But if $w = x + y + z$ and $z = x + y$ then the expression on the left hand side is $1 \cdot 1 = 1$, so that $0 = 1$. Where is the mistake?

9. Let α and β be norms on $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively. Prove that $\gamma_0(x, y) = \alpha(x) + \beta(y)$ and $\gamma_1(x, y) = \max(\alpha(x), \beta(y))$ define norms on $\mathbf{R}^{m+n} \cong \mathbf{R}^m \times \mathbf{R}^n$.

10. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a $1-1$ linear mapping. Prove that there is an $\varepsilon > 0$ such that if $S : \mathbf{R}^n \to \mathbf{R}^m$ is linear and satisfies $\|S - T\| < \varepsilon$, then S is also $1-1$.

11. Let $1 \leq r \leq \infty$.

(*i*) If U is open in $\mathbf{R}^n$, prove that the identity map id_U is a $\mathcal{C}^\infty$ diffeomorphism.

(*ii*) If U and V are open in $\mathbf{R}^n$ and $f : U \to V$ is a $\mathcal{C}^r$-diffeomorphism, then so is f^{-1}.

(*iii*) If U, V and W are open in $\mathbf{R}^n$, and $f : U \to V$ and $g : V \to W$ are $\mathcal{C}^r$ diffeomorphisms, then so is $g \circ f$.

12. (*i*) Suppose that X and Y are subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively and that $f : X \to Y$ and $g : Y \to \mathbf{R}^p$ are maps that satisfy Lipschitz conditions. Prove that the composite $g \circ f$ also satisfies a Lipschitz condition.

(*ii*) Suppose that $X \subset \mathbf{R}^n$, and let $f, g : X \to \mathbf{R}^m$ and $h : X \to \mathbf{R}$ satisfy Lipschitz conditions. Prove that $f + g$ satisfies a Lipschitz condition and if X is compact then $h \cdot f$ also satisfies a Lipschitz condition. If $h > 0$ and X is compact, does $1/h$ satisfy a Lipschitz condition? Prove this or give a counterexample.

(*iii*) Suppose that $X \subset \mathbf{R}^n$, and let $f : X \to \mathbf{R}^m$ be given. Prove that f satisfies a Lipschitz condition if and only if all of its coordinate functions do.

13. In the notation of the preceding exercise, suppose that $X = A \cup B$ and that f is continuous and satisfies Lipschitz conditions on A and B as well as on an open neighborhood of

$A \cap B$. Does f satisfy a Lipschitz condition on $A \cup B$? Prove this or give a counterexample. What happens if we assume A and B are compact? Justify your answer.

II.2 : Implicit and Inverse Function Theorems

(Conlon, Appendix B, §§ 2.4–2.5)

Additional exercises

1. Show that

$$f(x,y) \quad = \quad \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

is locally invertible near every point except the origin. Compute the inverse explicitly.

2. Consider the map : $\mathbf{R}^3 \to \mathbf{R}^3$ defined by $f(x,y,z) = (x, y^3, z^5)$. Note that f has a global inverse g despite the fact that $Df(0)$ is not invertible. What does this imply about the differentiability of g at 0?

3. Show that the mapping $(u,v,w) : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $u = x + e^y$, $v = y + e^z$ and $w = z + e^x$ is everywhere locally invertible.

4. Let $f : \mathbf{R}^3_{\mathbf{x}} \to \mathbf{R}^3_{\mathbf{y}}$ and $g : \mathbf{R}^3_{\mathbf{y}} \to \mathbf{R}^3_{\mathbf{x}}$ be $\mathcal{C}^1$ inverse functions. Show that

$$\frac{\partial g_1}{\partial y_1} \quad = \quad \frac{1}{J}\,\frac{\partial(f_2,f_3)}{\partial(x_2,x_3)}, \qquad J = \frac{\partial(f_1,f_2,f_3)}{\partial(x_1,x_2,x_3)}$$

and obtain similar formulas for the other derivatives of coordinate functions of g.

5. Prove that $F(x,y) = (e^x + y,\, x - y)$ defines a $\mathcal{C}^\infty$ homeomorphism of $\mathbf{R}^2$ with a $\mathcal{C}^\infty$ inverse.

6. Prove that $F(x,y) = (xe^y + y,\, xe^y - y)$ defines a $\mathcal{C}^\infty$ homeomorphism of $\mathbf{R}^2$ with a $\mathcal{C}^\infty$ inverse.

7. Prove that

$$F(x,y,z) \quad = \quad \left(\frac{x}{2+y^2} + ye^z, \frac{x}{2+y^2} - ye^z, 2ye^z + z\right)$$

defines a $\mathcal{C}^\infty$ homeomorphism of $\mathbf{R}^3$ with a $\mathcal{C}^\infty$ inverse.

8. Let $f(x,y) = (x+y, x^2+y)$. Check that f meets the conditions to have a local inverse near $f(1,0) = (1,1)$, and if g is this local inverse find $Dg(1,1)$ without finding a formula for the inverse function explicitly.

9. Consider the mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$ given by $f(x,y) = (x^2+y^2, 2xy)$. Show that the Jacobian vanishes on the lines $y = \pm\, x$. What is the image of f? [*Hint:* Try using polar coordinates.] The Inverse Function Theorem guarantees that f has a local inverse at $f(1,0) = (1,0)$. Find the inverse explicitly and describe a region on which it is defined.

10. The following example shows why it is necessary to assume the continuity at a point in the Inverse Function Theorem. Let $f(t) \;=\; t \,+\, 2t^2 \sin\left(\frac{1}{t}\right)$ for $t \neq 0$ and set $f(0) = 0$. Prove that $f'(0) = 1$, f' is bounded on $(-1,1)$, but f is not 1–1 on any neighborhood of 0.

11. (*i*) Let W be open in $\mathbf{R}^n$, and let $h : W \to \mathbf{R}^k$ be continuous. Prove that h is smooth if and only if there is an open covering $\mathcal{V}$ of W such that for each V_α in $\mathcal{V}$ the restriction $f|V_\alpha$ is smooth. (*ii*) Let U and V be open in $\mathbf{R}^n$, let $f : U \to V$ be a smooth surjective immersion/submersion, and suppose that $g : V \to \mathbf{R}^q$ is a continuous map such that $g \circ f$ is smooth. Prove that g is also smooth.

12. A continuous map $f : A \to X$ is a *retract* if there is a continuous map $g : X \to A$ such that $g \circ f = \mathrm{id}_A$. Suppose that A and X are open subsets of Euclidean spaces and f and g are smooth. Prove that f is an immersion.

II.3 : Bump functions

(Conlon, §2.6)

Additional exercises

1. Let $f : \mathbf{R} \to \mathbf{R}$ be a smooth function of class $\mathcal{C}^r$ where $1 \leq r \leq \infty$, and let K be a compact subset of $\mathbf{R}$. Then there is a smooth $\mathcal{C}^r$ function $g : \mathbf{R} \to \mathbf{R}$ such that $g|K = f|K$ and g vanishes off some compact set K' containing K. [*Hint:* We can take $K = [-a, a]$ and $K' = [-b, b]$ for some a, b such that $0 < a < b$.]

2. Suppose that $A \subset \mathbf{R}^n$ and $f : A \to \mathbf{R}^n$ is continuous. Suppose further that for each $a \in A$ there is an open neighborhood V_a of a such that $f|A \cap V_a$ extends to a smmoth function on V_a. Prove that there is an open set W containing A and a smooth function $g : W \to \mathbf{R}$ such that $g|A = f$. [*Hint:* Start with a locally refinement $\mathcal{U}$ of $\mathcal{V} = \{ V_a \}$ and a partition of unity subordinate to $\mathcal{U}$.]

3. The following exercise will be based upon an important result for uniform convergence of infinite series to a differentiable function that follows from a more general result: Theorem 7.17 on pp.152–153 of BABY RUDIN: *Suppose we are given a sequence of uniformly absolutely convergent smooth $\mathcal{C}^1$ functions $\{ f_n \}$ on an interval U such that $\sum_n f'_n$ also converges uniformly and absolutely. Then f is a smooth function and $f' = \sum_n f'_n$.*

(*i*) Explain why this result generalizes to smooth $\mathcal{C}^1$ functions defined on an open set $U \subset \mathbf{R}^q$ for some $q \geq 0$ with f'_n replaced by ∇f_n (and vector length replacing the absolute value of a real number).

(*ii*) Let $U \subset \mathbf{R}^q$ be open, and let F be a closed subset of U. Prove that there is a smooth $\mathcal{C}^1$ function $h : U \to \mathbf{R}$ such that for all $x \in U$ we have $h(x) = 0 \Longleftrightarrow x \in F$; *i.e.*, in analogy with a result about continuous functions on metric spaces, every closed subset of U is the zero set for some smooth $\mathcal{C}^1$ function on U. [*Hints:* Take the usual sort of locally finite countable open covering of $U - F$ by ordinary open disks such that shrunken disks of half the radius still cover $U - F$, and let g_k be the smooth function defined on the k^{th} disk using a bump function, where as usual g_k extends smoothly to all of U by setting it equal to zero off the disk. Choose positive constants M_k such that $|g_k|$ and $|\nabla g_k|$ are both bounded from above by M_k, and set

$$h \quad = \quad \sum_n \frac{g_k}{M_k \cdot 2^k} \ .$$

Explain why h is a smooth $\mathcal{C}^1$ function and the zero set of h is equal to F.]

II.4 : Vector fields and integral flows

(Conlon, §§2.7–2.8, Appendix C.1–C.3)

Additional exercises

1. Find the flow associated to the vector field on $\mathbf{R}^2$ given by

$$y\frac{\partial}{\partial x} - y^3\frac{\partial}{\partial y}.$$

2. Find the flow associated to the vector field on $\mathbf{R}^3$ given by

$$ay\frac{\partial}{\partial x} - ax\frac{\partial}{\partial y} + a^2\frac{\partial}{\partial z}.$$

3. Find the flow associated to the vector field on $\mathbf{R}^3$ given by

$$y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

4. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation that has a basis of eigenvectors $\{\mathbf{v}_j\}$ with associated eigenvalues λ_j. Given a vector $\mathbf{x} \in \mathbf{R}^n$, express $\mathbf{x}$ as a linear combination $\sum_j c_j\,\mathbf{x}_j$. Verify that

$$\gamma(t) \quad = \quad \sum_j c_j \, \exp(\lambda_j\, t)\, \mathbf{v}_j$$

is a solution to the differential equation $\mathbf{y}' = T(\mathbf{y})$ with initial condition $\mathbf{x}(0) = \mathbf{x}$.

5. Show that the differential equation $y' = y^{2/3}$ with initial condition $y(0) = 0$ has infinitely many solutions. [*Hint:* Consider the functions y such that $y(t) = 0$ for $t \leq a$ and $y(t) = t^3$ for $t \geq a$. Some care is needed to compute the derivative of this function at $t = a$.]

6. Here is a slightly different application of the Contraction Lemma to a boundary value problem in the theory of differential equations.

(*i*) Suppose that $F : [a,b] \times \mathbf{R} \to \mathbf{R}$ is Lipschitz and K is a Lipschitz constant for F. Define a *Green's function* $G : [a,b] \times [a,b] \to \mathbf{R}$ by setting

$$G(s,t) \quad = \quad \begin{cases} \frac{(t-a)(b-s)}{(b-a)} & t \leq s \\ \frac{(s-a)(b-t)}{(b-a)} & s \leq t\,. \end{cases}$$

Note that this function is discontinuous on the diagonal but still integrable. Verify that a continuous function y on $[a,b]$ satisfies $y(t) = \int_a^b G(t,s)\,F(s,\,y(s))\,ds$ if and only if it satisfies the boundary value problem $y'' + F(t,y) = 0$, $y(a) = y(b) = 0$.

(*ii*) Show that $\int_a^b |G(t,s)|\,ds \leq (b-a)^2/4$.

(*iii*) Show that if $b - a$ is so small that $K(b-a)^2/4 < 1$, then there is a unique solution to the boundargy value problem $y'' + F(t,y) = 0$, $y(a) = y(b) = 0$. [*Hint:* Define T by $T\varphi(t) = \int_a^b G(t,s)\,F(t,\,\varphi(t))\,ds$ and show that T satisfies the hypothesis of the Contraction Lemma.]

III. Global theory of smooth manifolds and mappings

III.1 : Basic definitions and examples

(Conlon, §§3.1–3.2, 3.5)

Additional exercises

1. Let $\mathcal{E}$ be the smooth $\mathcal{C}^\infty$ atlas on $\mathbf{R}$ whose only chart is the identity, let $h : \mathbf{R} \to \mathbf{R}$ be the map defined by $h(x) = x^3$, and let $\mathcal{E}_h$ be the $\mathcal{C}^\infty$ atlas for $\mathbf{R}$ whose sole chart is $(\mathbf{R}, h)$. Prove that the map $h : (\mathbf{R}, \mathcal{E}) \to (\mathbf{R}, \mathcal{E}_h)$ is a diffeomorphism even though $h : (\mathbf{R}, \mathcal{E}) \to (\mathbf{R}, \mathcal{E})$ is a smooth map whose inverse is not smooth. — Generalize this result to an arbitrary continuous map $(U, \mathcal{E}) \to (U, \mathcal{E}_h)$ where h is open in $\mathbf{R}^n$ and h is a homeomorphism from U to itself.

2. Let $1 \leq s < r \leq \infty$, and let $(M, \mathcal{A})$ be a smooth $\mathcal{C}^r$ manifold where $\mathcal{A}$ is the maximal the notes we stated that $\mathcal{A}$ is also a $\mathcal{C}^s$ atlas but not a maximal $\mathcal{C}^s$-atlas. Proof the second part of this assertion. [*Hint:* There is a smooth $\mathcal{C}^s$ diffeomorphism of $\mathbf{R}^n$ that is not a smooth $\mathcal{C}^r$ diffeomorphism by results in the notes. Why is the analogous statement true if $\mathbf{R}^n$ is replaced by an open disk in $\mathbf{R}^n$? Use this to add extra charts to $\mathcal{A}$ such that the larger object is still a smooth $\mathcal{C}^s$ atlas.]

3. This exercise asks for a verification of a statement in the discussion of lens spaces. We recall the basic setting: Given a finite cyclic group $\mathbf{Z}_k$ of order k, and a positive integer n, let $(m_1, \cdots, m_n)$ be an ordered n-tuple of positive integers less than k such that each m_j is prime to k. Then a topological action of $\mathbf{Z}_k$ on $S^{2n-1} \subset \mathbf{C}^n \cong \mathbf{R}^{2n}$ is defined by the formula

$$g^j(z_1, \cdots, z_n \quad = \quad (\alpha^{m_1} z_1, \cdots, \alpha^{m_n} z_n)$$

where g denotes a standard generator of $\mathbf{Z}_k$ and $\alpha = \exp(2\pi i/k)$. Prove that this is a free action on $\mathbf{C}^n - \{\mathbf{0}\}$; *i.e.*, $g^j\mathbf{z} \neq \mathbf{z}$ if $j \not\equiv 0(k)$ and $\mathbf{z} \neq \mathbf{0}$.

4. (The **mapping torus** construction) Let X be a Hausdorff topological space, let $f : X \to X$ be a homeomorphism, and consider the regular quotient space

$$T_f = X \times \mathbf{R}/\mu_f$$

where μ_f is the equivalence relation $(x, s) \sim (y, t)$ if and only if there is an integer n such that $s = t + n$ and $y = f^n(x)$. The quotient space is called the *mapping torus* of f.

(*i*) If f is the identity map, prove that X_f is homeomorphic to $X \times S^1$.

(*ii*) Prove that T_f is Hausdorff. [*Hint:* First show that there is a well-defined continuous map q from T_f to S^1 taking the equivalence class of (x, t) to $\exp(2\pi i t)$. Suppose $u \neq v$ in T_f. If $q(u) \neq q(v)$ then there are disjoint open neighborhoods U and V of these points in S^1, and their inverse images $q^{-1}(U)$ and $q^{-1})(V)$ are disjoint open neighborhoods of u and v. On the other hand, if $q(u) = q(v) = z$ and W is the open semicircular arc centered at z, then $q^{-1}(W)$ is homeomorphic to $(-1, 1) \times M$, which is Hausdorff.]

(*iii*) Perhaps the simplest nontrivial example of this is the Klein bottle, for which $M = S^1$ and f is complex conjugation. Prove that there is a 2-sheeted regular covering map from T^2 to the Klein bottle.

(*iv*) Prove that the quotient space projection is a regular covering space projection for which the group of covering transformations is the infinite cyclic group generated by $\varphi(x, t) = (f(x), t+1)$.

(*v*) If M is a smooth manifold and f is a diffeomorphism, explain why M_f has a naturally associated smooth structure with an infinite cyclic group of covering diffeomorphisms.

(*vi*) Suppose that X is connected with fundamental group G, and suppose that f is basepoint preserving. Show that T_f has a fundamental group Γ such that G is a normal subgroup, Γ/G is infinite cyclic, and there is a generator γ for the quotient group such that $\gamma g \gamma^{-1} = f_*(g)$, where f_* is the automorphism of the fundamental group defined by the homeomorphism f.

(*vii*) Prove that the symmetric group on three letters is a quotient group of the fundamental group of the Klein bottle. Is the same statement true for the symmetry group of a regular n-gon in the plane? Prove this or give a counterexample.

5. Let M and N be smooth manifolds, and let $h : M \to N$ be a continuous mapping. Prove that h is smooth if and only if for each open subset $V \subset N$ and each smooth function $f : V \to \mathbf{R}$ the composite "$f \circ h$": $h^{-1}(V) \to \mathbf{R}$ is smooth.

6. Let $F : \mathbf{R}^4 \to \mathbf{R}^2$ be the smooth map defined by

$$F(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + 2 + y).$$

Show that $(0, 1)$ is a regular value and that the level set is diffeomorphic to S^2.

III.2 : Constructions on smooth manifolds

(Conlon, §§1.7, 3.7)

Additional exercises

1. (*i*) Let X, Y and Z be smooth manifolds and let $\times$ denote the usual cartesian product. Prove that

$$X \times (Y \times Z)$$

is a direct product of smooth manifolds as defined in the notes.

(*ii*) Let A, B, C and D be smooth manifolds and let $\times$ denote the usual cartesian product. Prove that

$$(A \times B) \ \times \ (C \times D)$$

is a direct product of smooth manifolds as defined in the notes.

[*Note:* These may all be viewed as special cases of a more general result.]

2. Let X and Y be smooth manifolds and let $\tau : X \times Y \to Y \times X$ be the "twist map" which sends (x, y) to (y, x) for all x and y. Prove that τ is a diffeomorphism. [*Hint:* Consider the analogous map $\tau' : Y \times X \to X \times Y$.]

(*ii*) Let X be a smooth manifold and let $T : X \times X \times X \to X \times X \times X$ be the map that cyclically permutes the coordinates: $T(x, y, z) = (z, x, y)$. Prove that T is a diffeomorphism. [*Hint:* What is the test for smoothness of a map into a product? Can you write down an explicit formula for the inverse function?]

3. ("A product of products is a product.") Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a finite family of smooth manifolds, and let $\mathcal{A} = \cup\{\mathcal{A}_\beta \mid \beta \in \mathcal{B}\}$ be a partition of $\mathcal{A}$. Construct a diffeomorphism from $\prod\{A_\alpha \mid \alpha \in \mathcal{A}\}$ to the set

$$\prod_\beta \{\prod\{A_\alpha \mid \alpha \in \mathcal{A}_\beta\}\ \} \ .$$

4. Let $\mathcal{A}$ be some nonempty set, let $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ and $\{Y_\alpha \mid \alpha \in \mathcal{A}\}$ be finite families of smooth manifolds, and for each $\alpha \in \mathcal{A}$ suppose that $f_\alpha : X_\alpha \to Y_\alpha$ is a diffeomorphism. Prove that the product map

$$\prod_\alpha f_\alpha \ : \ \prod_\alpha X_\alpha \longrightarrow \prod_\alpha Y_\alpha$$

is also a diffeomorphism. [*Hint:* What happens when you take the product of the inverse maps?]

5. Prove that $\mathbf{R}^n - \{0\}$ is diffeomorphic to $S^{n-1} \times \mathbf{R}$.

III.3 : Smooth approximations

(Conlon, §§3.5, 3.8)

Conlon, pp. 116–117: 3.8.3, 3.8.5, 3.8.6

Additional exercises

1. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a diffeomorphism.

(i) Why is the derivative f' always positive or always negative?

(ii) Prove that f is smoothly isotopic to the identity if f' is always positive and smoothly isotopic to minus the identity if f' is always negative. [*Hints:* It will simplify things to note first that one can find a diffeomorphism isotopic to f such that $f(0) = 0$. If $f' > 0$, what can one say about the straight line homotopy from f to the identity?]

(iii) Prove that every diffeomorphism of S^1 to itself is smoothly isotopic to either the identity or complex conjugation.

2. Let M be a smooth manifold. Two diffeomorphisms f and g from M to itself are said to be smoothly *concordant* or *pseudo-isotopic* if there is a homeomorphism H from $M \times [0,1]$ to itself with the following properties:

(1) The homeomorphism sends $M \times \{0\}$ to itself by f and $M \times \{1\}$ to itself by g.

(2) The homeomorphism is a diffeomorphism on $M \times (0,1)$.

(3) For each $x \in M$ there is an open neighborhood U and an $\varepsilon > 0$ such that the restrictions of H to $U \times [0,\varepsilon)$ and $U \times (\varepsilon, 1]$ depend only on the first variable. (If M is compact this is equivalent to saying that H is given by f on some open set of the form $M \times [0,\delta)$ and by g on some open set of the form $M \times (1-\delta, 1]$.)

Prove that concordance defines an equivalence relation on diffeomorphisms of M and that isotopic diffeomorphisms are concordant. [The difference is that a concordance does not send the level submanifolds $M \times \{t\}$ into themselves. Determining the relation between concordance and isotopy is a deep and difficult question that was essentially answered in the nineteen seventies by A. Hatcher and J. Wagoner for manifolds of sufficiently large dimension.]

III.4 : Amalgamation theorems

(Conlon, §1.3)

Additional exercises

1. Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of topological spaces, and let $X = \coprod_\alpha A_\alpha$. Prove that X is locally connected if and only if each A_α is locally connected.

2. In the preceding exercise, formulate and prove necessary and sufficient conditions on $\mathcal{A}$ and the sets A_α for the space X to be compact.

3. Prove that $\mathbf{RP}^2$ can be constructed by identifying the edge of a Möbius strip with the edge circle on a closed 2-dimensional disk by filling in the details of the following argument: Let $A \subset S^2$ be the set of all points $(x, y, z) \in S^2$ such that $|z| \leq \frac{1}{2}$, and let B be the set of all points where $|z| \geq \frac{1}{2}$. If $T(x) = -x$, then $T(A) = A$ and $T(B) = B$ so that each of A and B (as well as their intersection) can be viewed as a union of equivalence classes for the equivalence relation that produces $\mathbf{RP}^2$. By construction B is a disjoint union of two pieces $B_\pm$ consisting of all points where $\mathrm{sign}(z) = \pm 1$, and thus it follows that the image of B in the quotient space is homeomorphic to $B_+ \cong D^2$. Now consider A. There is a homeomorphism h from $S^1 \times [-1, 1]$ to A sending (x, y, t) to $(\alpha(t)x, \alpha(t)y, \frac{1}{2}t)$ where

$$\alpha(t) = \sqrt{1 - \frac{t^2}{4}}$$

and by construction $h(-v) = -h(v)$. The image of A in the quotient space is thus the quotient of $S^1 \times [-1, 1]$ modulo the equivalence relation $u \sim v \iff u = \pm v$. This quotient space is in turn homeomorphic to the quotient space of the upper semicircular arc S^1_+ (all points with nonnegative y-coordinate) modulo the equivalence relation generated by setting $(-1, 0, t)$ equivalent to $(1, 0, -t)$, which yields the Möbius strip. The intersection of this subset in the quotient with the image of B is just the image of the closed curve on the edge of B_+, which also represents the edge curve on the Möbius strip.

4. Suppose that the topological space X is a union of two closed subspaces A and B, let $C = A \cap B$, let $h : C \to C$ be a homeomorphism, and let $A \cup_h B$ be the space formed from $A \sqcup B$ by identifying $x \in C \subset A$ with $h(x) \in C \subset B$. Prove that $A \cup_h B$ is homeomorphic to X if h extends to a homeomorphism $H : A \to A$, and give an example for which X is not homeomorphic to $A \cup_h B$. [*Hint:* Construct the homeomorphism using H in the first case, and consider also the case where $X = S^1 \sqcup S^1$, with $A_\pm == S^1_\pm \sqcup S^1_\pm$; then $C = \{\pm 1\} \times \{1, 2\}$, and there is a homeomorphism from h to itself such that $A_+ \cup_h A_-$ is connected.]

5. [*One-point unions.*] One conceptual problem with the disjoint union of topological spaces is that it is never connected except for the trivial case of one summand. In many geometrical and topological contexts it is extremely useful to construct a modified version of disjoint unions that is connected if all the pieces are. Usually some additional structure is needed in order to make such constructions.

In this exercise we shall describe such a construction for objects known as *pointed spaces* that are indispensable for many purposes (*e.g.*, the definition of fundamental groups as in Munkres). A pointed space is a pair (X, x) consisting of a topological space X and a point $x \in X$; we often call x the *base point*, and unless stated otherwise *the one point set consisting of the base point is assumed to be closed.* If (Y, y) is another pointed space and $f : X \to Y$ is continuous, we shall say that f *is a base point preserving continuous map from* (X, x) *to* (Y, y) if $f(x) = y$, In this case we shall often write $f : (X, x) \to (Y, y)$. Identity maps are base point preserving, and composites of base point preserving maps are also base point preserving.

(i) Given a finite collection of pointed spaces (X_i, x_i), define an equivalence relation on $\coprod_i X_i$ whose equivalence classes consist of $\coprod_j \{x_j\}$ and all one point sets y such that $y \notin \coprod_j \{x_j\}$. Define the *one point union* or *wedge*

$$\bigvee_{i=1}^{n} (X_j, x_j) = (X_1, x_1) \vee \cdots \vee (X_n, x_n)$$

to be the quotient space of this equivalence relation with the quotient topology. The base point of this space is taken to be the class of $\coprod_j \{x_j\}$.

(i) Prove that the wedge is a union of closed subspaces Y_j such that each Y_j is homeomorphic to X_j and if $j \neq k$ then $Y_j \cap Y_k$ is the base point. Explain why $\vee_k\ (X_k, x_k)$ is Hausdorff if and only if each X_j is Hausdorff, why $\vee_k\ (X_k, x_k)$ is compact if and only if each X_j is compact, and why $\vee_k\ (X_k, x_k)$ is connected if and only if each X_j is connected (and the same holds for arcwise connectedness).

(ii) Let $\varphi_j : (X_j, x_j) \to \vee_k\ (X_k, x_k)$ be the composite of the injection $X_j \to \coprod_k\ X_k$ with the quotient projection; by construction φ_j is base point preserving. Suppose that (Y, y) is some arbitrary pointed space and we are given a sequence of base point preserving continuous maps $F_j : (X_j, x_j) \to (Y, y)$. Prove that there is a unique base point preserving continuous mapping

$$F : \vee_k\ (X_k, x_k) \to (Y, y)$$

such that $F \circ \varphi_j = F_j$ for all j.

(iii) In the infinite case one can carry out the set-theoretic construction as above but some care is needed in defining the topology. Show that if each X_j is Hausdorff and one takes the so-called *weak topology* whose closed subsets are generated by the family of subsets $\varphi_j(F)$ where F is closed in X_j for some j, then [1] a function h from the wedge into some other space Y is continuous if and only if each composite $h \circ \varphi_j$ is continuous, [2] the existence and uniqueness theorem for mappings from the wedge (in the previous portion of the exercise) generalizes to infinite wedges with the so-called weak topologies.

(iv) Suppose that we are given an infinite wedge such that each summand is Hausdorff and contains at least two points. Prove that the wedge with the so-called weak topology is not compact.

Remark. If each of the summands in (iv) is compact Hausdorff, then there is a natural candidate for a *strong topology* on a countably infinite wedge which makes the latter into a compact Hausdorff space. In some cases this topology can be viewed more geometrically; for example, if each (X_j, x_j) is equal to $(S^1, 1)$ and there are countably infinitely many of them, then the space one obtains is the *Hawaiian earring* in $\mathbf{R}^2$ given by the union of the circles defined by the equations

$$\left(x - \frac{1}{2^k}\right)^2 + y^2 = \frac{1}{2^{2k}} .$$

As usual, drawing a picture may be helpful. The k^{th} circle has center $(1/2^k,\ 0)$ and passes through the origin; the y-axis is the tangent line to each circle at the origin.

6. Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of topological spaces, and let $X = \coprod_\alpha A_\alpha$. Formulate and prove necessary and sufficient conditions on $\mathcal{A}$ and the sets A_α for the space X to be second countable, separable or Lindelöf.

III.5 : Tangent spaces and vector bundles

(Conlon, §§3.3–3.4)

Additional exercises

1. Prove the following results on tangent bundles.

(*i*) $T(\mathbf{R}^n) \cong \mathbf{R}^n \times \mathbf{R}^n$ such that τ corresponds to projection onto the first factor and the vector space operations on $\{pt.\} \times \mathbf{R}^n$ are given by the standard $1-1$ correspondence between the latter and $\mathbf{R}^n$.

(*ii*) If M and N are smooth manifolds, then $T(M \times N) \cong T(M) \times T(N)$ such that $\tau_{M \times N}$ correspond to $\tau_m \times \tau_N$.

(*iii*) If P is a smooth manifold and V is an open subset of P, then $T(V) \cong \tau_M^{-1}(V)$ such that τ_V corresponds to $\tau_M | T(V)$.

2. Let M be a smooth manifold, let $T(M)$ be its tangent space, and let $z : T(M) \to T(M)$ be the map sending each tangent vector to the zero vector at the same point. Prove that z is smoothly homotopic to the identity.

3. Suppose that $f : M \to N$ is a a smooth homeomorphism. Prove that f is a diffeomorphism if and only if $T(f)$ is 1–1 and onto.

III.6 : Regular mappings and submanifolds

(Conlon, §§1.5, 2.5, 3.7)

Conlon, p. 62: 2.5.15(2)

Additional exercises

A: EXERCISES ON IMMERSIONS AND SUBMERSIONS

1. (*i*) Let $f : M \to N$ be a smooth homoeomorphism of smooth manifolds. Then f is a diffeomorphism if and only if f is an immersion.

(*ii*) Let $f : M \to N$ be a smooth homoeomorphism of smooth manifolds. Then f is a diffeomorphism if and only if f is a submersion.

(*iii*) Let $f : M \to N$ and $g : N \to P$ be smooth mappings of smooth manifolds. If f and g are immersions, then so is their composite $g \circ f$.

(*iv*) Let $f : M \to N$ and $g : N \to P$ be smooth mappings of smooth manifolds. If f and g are submersions, then so is their composite $g \circ f$.

2. Prove that a smooth map from the 2-sphere to the unit circle cannot be 1–1.

3. Prove that there is no immersion from a compact n-manifold into $\mathbf{R}^n$.

4. Given an immersion from a 1-connected compact smooth manifold onto a smooth manifold of the same dimension, prove that it is a covering projection. Does the statement remain true if the manifolds are not necessarily compact? Prove this or give a counterexample.

5. Let $p : E \to B$ be a topological or smooth fiber bundle projection, and assume that p is onto and B is connected. Prove that the homeomorphism or diffeomorphism type of $p^{-1}(\{x\})$ is the same for all $x \in B$. [*Hint:* If we say $x \sim y$ if and only if the inverse images of these points are homeomorphic or diffeomoorphic, then $\sim$ is an equivalence relation and it is locally constant.]

6. (*i*) Suppose that M and N are smooth manifolds. A smooth map $f : M \to N$ is said to be a *retract* if there is a smooth map $g : N \to M$ such that $g \circ f = \mathrm{id}_M$. Prove that a smooth retract is a smooth immersion.

(*ii*) A smooth map of smooth manifolds $r : N \to M$ is said to be a smooth *retraction* if there is a smooth map $j : M \to N$ such that $r \circ j = \mathrm{id}_M$. Prove that if r is a retraction, then the restriction of r to some neighborhood of $j(M)$ is a submersion.

(*iii*) A continuous map of topological spaces is said to be a continuous retract if it satisfies the condition in (*i*). Prove that if A and X are Hausdorff then j is a closed mapping. [*Hint:* To see that A is closed, show that it is the set of all points such that $x = j \circ r(x)$.]

B: EXERCISES ON SUBMANIFOLDS AND EMBEDDINGS

1. Suppose that U is open in $\mathbf{R}^n$ and that $f : U \to \mathbf{R}^n$ and $g : U \to \mathbf{R}^m$ are smooth functions where $m < n$. Let $x \in U$ be a point on the level set L on which $g(x) = 0$, and suppose that $Dg(x)$ has rank m ($\Longrightarrow$ if we restrict to a suitable open neighborhood V of x in U, the set $L \cap V$ is a smooth submanifold of dimension $n - m$).

(*i*) Suppose that $f|L$ has a local maximum at x. Prove that $\nabla f(x)$ is perpendicular to the tangent space $T_x(L)$. [*Hint:* What can we say about $D[f|L](x)$ under the given hypothesis?]

(*ii*) If the coordinates of g are given by g_j (where $1 \le j \le m$), explain why the orthogonal complement of $T_x(L)$ is spanned by the vectors $\nabla g_j(x)$.

(*iii*) Using the preceding parts of this exercise, derive the **Lagrange Multiplier Rule**: One can find m constants (or Lagrange multipliers) λ_j such that $\nabla f(x) = -\sum_j \nabla g_j(x)$ or equivalently x (and the λ_j's) determine a solution to the following system of equations:

$$\nabla \left(f \;+\; \sum_j \lambda_j \, g_j \right) \;=\; \mathbf{0}$$

$$g(x) = 0$$

Note that this is a system of $m+n$ scalar equations in the n coordinates of x and the m multipliers λ_j.

2. Let $f(x,y) = x^3 + xy - x^3$. Show that the level set for the value 1 is a smooth submanifold but the level set for the value 0 is not. [*Hint:* In the second case, prove that otherwise there would be a pair of C^∞ functions x and y satisfying $f(x,y) = 0$ where $x(0) = y(0) = 0$ and $(x'(0), y"(0)) \neq (0,0)$. Then consider the existence of the limits of $x(t)/y(t)$ and $y(t)/x(t)$ as $t \to 0$.]

3. For which real numbers c is the set $y^2 - x(x-1)(x-c) = 0$ a submanifold of $\mathbf{R}^2$?

4. Let $f(x,y) = y^2 + x^4 - x^2$. Find all real numbers c such that the level set $f^{-1}(\{c\})$ is a smooth submanifold. Give reasons for your answer.

5. Let $Q \subset \mathbf{R}^{n+1}$ be the unit cube consisting of all $(x_0, \cdots x_n)$ such that $\max_i |x_i| = 1$. Prove that Q is homeomorphic to S^n, that Q has a smooth atlas for which Q is diffeomorphic to S^n, but Q is not a smooth submanifold of $\mathbf{R}^{n+1}$.

6. Let X be the y-axis in the Cartesian plane, and let Y be the graph of $\sin \frac{1}{x}$ for $x > 0$. Prove that $X \cup Y$ is an immersed but not embedded submanifold but that each of X and Y taken separately is an embedded submanifold.

7. Let A be a real nonsingular symmetric $n \times n$ matrix and let c be a nonzero real number. Show that the quadric hypersurface defined by the equation $\langle Ax, \ x \rangle = c$ is a smooth $n-1$-dimensional submanifold of $\mathbf{R}^n$.

8. Let M be a noncompact smooth manifold. Prove that there is a smooth embedding $f : (-\varepsilon, \infty) \to M$ such that the image of $[0, \infty)$ is a closed subset.

9. Suppose that M is a noncompact smooth manifold and there is a smooth 1–1 immersion $f : M \to \mathbf{R}^N$. Prove that there is a smooth embedding $g : M \to \mathbf{R}^{N+1}$ such that $g(M)$ is a closed subset.

10. Let M be a smooth submanifold of N that is a closed subset of N, and let f and X be a smooth real valued function and a smooth vector field on M respectively. Prove that f and X extend to smooth function and vector field (respectively) on N. [*Hint:* Use submanifold charts.]

11. Let $f : S^2 \to \mathbf{R}^4$ be the smooth map sending (x, y, z) to $(x^2 - y^2, xy, xz, yz)$. Show that $f(x, y, z) = f(-x, -y, -z)$ for all (x, y, z) and that the associated map $g : \mathbf{RP}^2 \to \mathbf{R}^4$ on the quotient manifold is a smooth embedding.

12. Show that it is possible to make the subset of the plane defined by the equation $x^3 - y^2 = 0$ into a smooth manifold but that the set in question is not a smooth submanifold of $\mathbf{R}^2$. What happens for the set $x^4 - y^2 = 0$?

13. Let $f : S^2 \to \mathbf{R}^4$ be the smooth map sending (x, y, z) to $(x^2 - y^2, xy, xz, yz)$. Show that $f(x, y, z) = f(-x, -y, -z)$ for all (x, y, z) and that the associated map $g : \mathbf{RP}^2 \to \mathbf{R}^4$ on the quotient manifold is a smooth embedding.

14. Let $A \subset \mathbf{R}^2$ be the graph of the function $f(t) = |t|$. Prove that A is a topologically locally flat submanifold of $\mathbf{R}^2$ but not a smooth submanifold. [*Hints:* Construct a homeomorphism from $\mathbf{R}^2$ to itself that sends A to the x-axis. To show A is not a smooth submanifold, derive a contradiction by finding two candidates for the tangent space at the origin.]

15. Let $n_1, \ \cdots \ , n_k$ be positive integers and let N be their sum. Prove that there is a smooth embedding of $\prod_j \ S^{n_j}$ into s^{N+1}. [*Hint:* One always has smooth embeddings of $S^p \times \mathbf{R}^q$ in $\mathbf{R}^{p+q}$ and embeddings of $S^{q-1} \times \mathbf{R}$ in $\mathbf{R}^q$. Use these as part of an inductive argument.]

16. Suppose we have smooth maps $i : M \to N$ and $j : N \to L$ such that $j \circ i$ is a smooth embedding. Prove that i is a smooth embedding.

17. Show that the set of all orthogonal $n \times n$ matrices is a compact submanifold of the group $GL(n, \mathbf{R})$ of all invertible matrices. [*Hint:* Show that the identity matrix I is a regular value of the function ${}^{\mathbf{T}}\!A \cdot A$ from $GL(n, \mathbf{R})$ to the vector space of all symmetric $n \times n$ matrices viewed as Euclidean space of dimension $\frac{1}{2}\,n(n+1)$.]

18. Consider the set $LF_{n,k}$ of *labeled flexible n-gons* in $\mathbf{R}^k$. These are the figures obtained by joining $n > 2$ straight line segments of unit length into a closed curve.

(i) Suppose that n is odd and $k = 2$. Prove that $LF_{n,2}$ is a smooth submanifold of $\mathbf{R}^2 \times T^{n-1}$ whose dimension is equal to n.

(ii) Prove that the set of all such objects with no self-intersections is a smooth manifold.

19. Let μ denote the standard Lebesgue measure on $\mathbf{R}^n$. Recall that a subset $A \subset \mathbf{R}^n$ **has (Lebesgue) measure zero** if for every $\varepsilon > 0$ there is a countable family of open subsets U_j such that $A \subset \cup_j\, U_j$ and $\sum_j\, \mu(U_j) \;<\; \varepsilon$. Also recall that a subset of a set with measure zero also has measure zero and countable union of sets with measure zero also has measure zero.

(i) Suppose that U and V are open subsets of $\mathbf{R}^n$ and $\varphi : U \to V$ is a $\mathcal{C}^1$ diffeomorphism. Prove that if $A \subset U$ has measure zero, then $\varphi(A)$ also has measure zero. [*Hint:* If we express U as an increasing union of the compact subsets K_n it suffices to show that each $\varphi(A \cap K_n)$ has measure zero. Recall that φ satisfies Lipschitz conditions on the sets K_n.]

(ii) Given a smooth manifold M, we shall say that a subset $A \subset M$ has measure zero if for all smooth charts (U, h) in a maximal atlas the inverse image $h^{-1}(A)$ has measure zero. — Prove that $A \subset M$ has measure zero if and only if there is a countable family of smooth charts (V_β, k_β) such that the images cover A and $k_\beta^{-1}(A)$ has measure zero for all α. [*Hint:* The $(\Longrightarrow)$ implication is trivial. To prove the reverse implication, let (U, h) be an arbitrary smooth chart, and for each β let $A'_\beta = h^{-1}(A \cap k_\beta(V_\beta)$. Why does the hypotheses for the $(\Longleftarrow)$ direction imply this set has measure zero? Finish off by checking that $h^{-1}(A) = \cup_\beta\, A'_\beta$.]

(iii) Given a set of measure zero in a smooth manifold M, show that it must be nowhere dense in M.

(iv) Let M and N be smooth manifolds such that $\dim M < \dim N$, and suppose that $f : M \to N$ is a smooth immersion. Prove that the image of f has measure zero ($\Longrightarrow$ is nowhere dense in M).

REMARK. It is well known the conclusion in (i) does **not** extend to homeomorphisms; it is possible (and in fact not very difficult) to construct a homeomorphism that takes a compact set of measure zero to a set of positive measure. Here is an online reference:

`http://www.cut-the-knot.org/do_you_know/Cantor2.shtml`

20. (i) Suppose that U is open in $\mathbf{R}^2$ and $\sigma : U \to \mathbf{R}^3$ is a smooth map such that the cross products of the partial derivatives satisfies

$$\frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \;\neq\; 0$$

on U. Explain why σ is a smooth immersion.

(ii) Explain why the cross product vector is always perpendicular to the image of $D\sigma$.

(iii) In the setting above let $\vec{N}$ denote the cross product of the partial derivative vectors and define a function of three variables by

$$\Theta(u, v, w) \;\;=\;\; \sigma(u, v) \;+\; w \cdot \vec{N}(u, v)\ .$$

For each $x = (u_0, v_0) \in U$ explain why one can find an open subneighborhood $U_0 \subset U$ containin x and an $\varepsilon > 0$ such that Θ maps $U_0 \times (-\varepsilon, \varepsilon)$ diffeomorphically onto an open neighborhood of $\sigma(x)$ in $\mathbf{R}^3$.

Still further exercises for Section III.4

We continue the previous numbering.

7. Prove that the disjoint union (or sum) topology on $\coprod_\alpha X_\alpha$ has the following basic properties:

(*i*) The family of subsets $\sum_\alpha \mathbf{T}_\alpha$ defines a topology for $\coprod_\alpha X_\alpha$ such that the injection maps i_α are homeomorphisms onto their respective images. the latter are open and closed subspaces of $\coprod_\alpha X_\alpha$, and each injection is continuous, open and closed.

(*ii*) The closed subsets of $\coprod X_\alpha$ with the disjoint union topology are the sets of the form $\coprod F_\alpha$ where F_α is closed in X_α for each α.

(*iii*) If each X_α is discrete then so is $\coprod_\alpha X_\alpha$.

(*iv*) If each X_α is Hausdorff then so is $\coprod_\alpha X_\alpha$.

(*v*) If each X_α is homeomorphic to a metric space, then so is $\coprod_\alpha X_\alpha$.

(*vi*) If for each α we are given a continuous function $f : X_\alpha \to W$ into some fixed space W, then there is a unique continuous map $h : \coprod_\alpha X_\alpha \to W$ such that $h \circ i_\alpha = f_\alpha$ for all α.

IV. Vector fields

IV.1 : Global vector fields

(Conlon, §§2.2, 3.3)

Additional exercises

1. Let X be the vector field on $\mathbf{R}^3$ such that $X(p) = (p : (1,1,1))$, and let

$$\mathbf{S} : (0,\infty) \times (0, 2\pi) \times (-\frac{1}{2}\,\pi,\ \frac{1}{2}\,\pi) \to \mathbf{R}^3 - \{(x,y,z) \mid x \ \leq \ 0 \quad \text{or} \quad x \ = \ y = \ 0\}$$

be the spherical coordinate diffeomorphism

$$(x,y,z) = (\rho \cos\theta \cos\phi,\ \rho \sin\theta \cos\phi, \rho \sin\phi).$$

What are the formulas for the coordinates of the vector field $\mathbf{S}_*^{-1}(X)$?

2. Construct a smooth vector field on S^2 that is zero at exactly one point. [*Hint:* Use stereographic projection.]

IV.2 : Global flows and completeness

(Conlon, §§2.7–2.8, 4.1)

Conlon, pp. 133–134: 4.1.9, 4.1.13

Additional exercises

1. Consider the nonautonomous differential equation

$$\frac{dx}{dt} \quad - \quad 2\,t$$

on the real line.

(i) Find the unique solution curve to this equation with initial condition $a \in \mathbf{R}$.

(ii) Let Φ be the flow map giving the solution curves to this equation. Show by example that

$$\Phi\big(s,\ \Phi(t,a)\big) \quad \neq \quad \Phi(s+t, a)$$

for suitably chosen s, t, a.

2. Consider the vector field Y on the plane defined by the vector-valued function (y, y^2). Find the integral curve φ_p of Y such that $\varphi(0) = p$, and specify the maximal interval for which this curve is defined. For which points in the plane does $\varphi_p(1)$ exist?

3. Let X be a smooth vector field on a manifold M, and let $f : M \to (0,\infty)$ be smooth. Show that the maximal integral curves $\varphi_p(t)$ for X and $\psi_p(t)$ for $f \cdot X$ with initial point p are reparametrizations of each other. If $(-a^-, a^+)$ and $(-b^-, b^+)$ are the maximal intervals on which φ_p and ψ_p are defined, show that

$$b^+(p) = \int_0^{a^+(p)} \frac{dt}{f(\phi_p(t))}$$

and

$$b^-(p) = \int_{-a^-(p)}^{0} \frac{dt}{f(\phi_p(t))}.$$

4. Given a smooth vector field on a noncompact connected (second countable) manifold M, show that there is a smooth function $f : M \to (0, \infty)$ such that $f \cdot X$ is complete. [*Hint:* Take an increasing family of compact subspaces K_i such that $K_i \subset \mathrm{Int}(K_{i+1})$ and $M = \cup_i K_i$. Note that

$$K_i - K_{i-1} \subset \mathrm{Int}(K_{i+1} - K_{i-1})$$

and therefore there exists a nonnegative smooth function ρ_i on M that is 1 on $K_i - K_{i-1}$ and whose support (= closure of the set where the function is nonzero) is contained in $\mathrm{Int}(K_{i+1} - K_{i-1})$. By convention K_0 and K_{-1} are empty. For each i show that there is an $\varepsilon_i > 0$ such that $|t| < \varepsilon_i \Rightarrow \Phi_t(K_i) \subset K_{i+1}$, where Φ denotes the flow of X. If

$$f = \sum_{i \geq 1} \varepsilon_i \cdot \rho_i$$

verify that f is a smooth positive function on M and use the preceding exercise to show that

$$b^+(p) = \int_0^{a^+(p)} \frac{dt}{f(\phi_p(t))} \geq \int_0^{\varepsilon_i} \frac{dt}{\varepsilon_i} = 1$$
$$b^-(p) = \int_{-a^-(p)}^{0} \frac{dt}{f(\phi_p(t))} \geq \int_{-\varepsilon_i}^{0} \frac{dt}{\varepsilon_i} = 1$$

where $a^\pm$ and $b^\pm$ are defined as in the preceding exercise. This means that the domain of the flow for $f \cdot X$ contains $(-1, 1) \times M$.]

5. Suppose that X and Y are smooth vector fields on an open set in some Euclidean space, and let D_X and D_Y be the corresponding derivations on the ring of smooth functions $C^\infty(M)$. Give an example to show that $D_X D_Y$ is not necessarily a derivation.

IV.3 : Lie brackets

(Conlon, §§2.2, 2.8, 4.3)

Conlon, p. 90: 2.7.19

Additional exercises

1. Let X and Y be the vector fields in the plane defined by the vector-valued smooth functions (x, xy) and (y^2, xy) respectively. Compute the Lie bracket $[X, Y]$.

2. Let Φ and Ψ be the 1-parameter groups of diffeomorphisms of $\mathbf{R}^3$ defined by clockwise rotation about the x and y axes respectively, and let A and B be the associated vector fields. Compute the Lie bracket $[A, B]$.

3. Find the Lie brackets of the following pairs of vector fields on $\mathbf{R}^3$ (we write ∂_u for $\frac{\partial}{\partial u}$ to save space):

(*i*) $y\,\partial_z - 2xy^2\,\partial_y$ and ∂_y.

(*ii*) $-y\,\partial_x + x\,\partial_y$ and $y\,\partial_x + x\,\partial_y$.

4. Suppose that a smooth function satisfies $[fX, Y] = f[X, Y]$ for all vector fields X and Y. What can one say about f?