Selected answers for

**Mathematics 205C, Spring 2003, Examination 1**

*Each problem is worth 25 points.*

1. $(i)$ If $M$ is a topological $n$-manifold and $x \in M$, then for each open neighborhood $U$ of $x$ there is an open subneighborhood $V \subset U$ such that $V$ is connected and $V - \{x\}$ has $C_n$ connected components, where $C_n$ is a positive integer that depends only on $n$. Give the exact values for the constants $C_n$.

$(ii)$ State the definitions of a smooth atlas for a topological manifold $M$ and of a smooth manifold.

**SOLUTION TO PART** $(i)$**:**

We have $C_1 = 2$ and $C_n = 1$ for $n \geq 2$. The correct value for $C_0$ is 0 because a 0-manifold is a discrete set (and therefore we can take $V = \{x\}$).

**SOLUTION TO PART** $(ii)$**:**

A smooth atlas for a topological $n$-manifold is a collection $\mathcal{A}$ of objects $(U_\alpha, h_\alpha)$ such that each $U_\alpha$ is open in $\mathbf{R}^n$, each $h_\alpha$ is a map from the corresponding $U_\alpha$ to $\mathbf{R}^n$ that is a homeomorphism onto an open subset, the image sets $h_\alpha(U_\alpha)$ form an open covering of $M$, and the transition maps "$h_\beta^{-1} \circ h_\alpha$" from $h_\alpha^{-1}(h_\beta(U_\beta))$ to $h_\beta^{-1}(h_\alpha(U_\alpha))$ are smooth ($C^\infty$) diffeomorphisms.

A smooth manifold is a pair $(M, \mathcal{A})$ consisting of a topological $n$-manifold $M$ and a *maximal* smooth atlas $\mathcal{A}$ for $M$.

2. Let $U$ and $V$ be open subsets of Euclidean spaces and let $q : U \to V$ be a smooth map. A smooth map $s : V \to U$ is said to be a *smooth cross section* of $q$ if the composite $q \circ s$ is the identity. Prove that a smooth cross section is always a 1–1 immersion.

**SOLUTION:**

First of all, $s$ is $1 - 1$ because $s(y) = s(z)$ implies $q(s(y)) = q(s(z))$ and since $q \circ s$ is the identity the second equation reduces to $y = z$.

To see that $s$ is an immersion, note that $qs = 1_V$ implies that

$$I = D(1_V) = D(q \circ s)$$

where $D$ denotes the derivative, so that the Chain Rule implies

$$I = Dq\,(s(y)) \circ Ds(y)$$

for all $y \in V$. By elementary linear algebra, this implies that the kernel $Ds(y)$ is always the zero subspace and that $Ds(y)$ is $1 - 1$.

**Note.** The map $q$ is a submersion on a neighborhood of $s(V)$ because $q$ has maximum rank on the latter the set of all points $u \in U$ for which $Dq(u)$ has maximum rank is always open. However, it need not be a submersion everywhere. Take $U = \mathbf{R}^2$, $V = \mathbf{R}$, $s(v) = (v, 0)$. Let $h$ be a smooth real valued funtion on the real line so that $h(t) = 1$ for $|t| \leq 1$ and $h(t) = 0$ for $|t| \geq 2$. Then the map $q(x, y) = (xh(y), yh(y))$ is not a submersion (it is zero for $|y| \geq 2$) but $qs$ is the identity.

3. Outline the construction of a smooth function $\varphi : \mathbf{R} \to \mathbf{R}$ such that $\varphi(x) = 0$ for $x \leq 0$, $\varphi$ is increasing for $x \in [0, 1]$, and $\varphi(x) = 1$ for $x \geq 1$.

**SOLUTION:**

The function $f(x) = \exp(-\frac{1}{x^2})$, which is defined for all real $x \neq 0$, has the property that

$$\lim_{x \to 0} f^{(n)}(x) = 0$$

for all nonnegative integers $n$, where as usual $f^{(n)}$ denotes the $n$-th derivative of $f$.

If we define $g$ by $g(x) = f(x)$ if $x > 0$ and $g(x) = 0$ if $x \leq 0$ then $g$ will be a $C^\infty$ function.

Define $h(x) = g(x) \cdot g(1 - x)$ to obtain a $C^\infty$ function that is positive if $x \in (0, 1)$ and zero otherwise.

Take $k(x)$ to be the (unique) antiderivative of $h$ such that $k(0) = 0$. Then $k'(x) = h(x) = 0$ for $x \leq 0$ implies that $k(x) = 0$ for $x \geq 0$, and $k'(x) = h(x) = 0$ for $x \leq 1$ implies that

$$k(x) = k(1) = \int_0^1 h(t)\,dt$$

for $x \geq 1$. Since $h$ is positive on (0,1), the antiderivative $k$ must be strictly increasing on that interval, and the integral of $h$ over the unit interval must be positive. Finally, take $\varphi(x) = k(x)/k(1)$.

4. Let $M$ and $N$ be smooth manifolds, suppose that $f$ and $g$ are diffeomorphisms from $M$ to $N$, and let $f \times g : M \times M \to N \times N$ be the product map. Given a smooth manifold $Y$, let $T_Y$ be the twist map on $Y \times Y$ defined by $T_Y(u, v) = (v, u)$. Prove that the composite $T_N \circ (f \times g) : M \times M \to N \times N$ is also a diffeomorphism. [*Hints:* Is $T_N$ a diffeomorphism? If so, what is its inverse? Consider $f^{-1} \times g^{-1}$ and the inverse function identity $(h \circ k)^{-1} = k^{-1} \circ h^{-1}$.]

**SOLUTION:**

The map $T_N$ is a diffeomorphism because it is its own inverse (*Proof:* $T_n(T_N(x, y)) = T_N(y, x) = (x, y)$). The map $f^{-1} \times g^{-1}$ is seen to be an inverse to $f \times g$ by direct calculation. In particular,

$$f^{-1} \times g^{-1}\,(f \times g(x, y)) = f^{-1} \times g^{-1}\,(f(x), g(y)) = \times g^{-1}\left(f^{-1}\,(f(x))\,, g^{-1}\,(g(y))\right) = (x, y)$$

for all $x$ and $y$, and similarly if we interchange $f$ and $g$ with $f^{-1}$ and $g^{-1}$ respectively.

Finally, the inverse formula shows that the composite of two diffeomorphisms is a diffeomorphism, and since $T_N \circ (f \times g)$ has been shown to be a composite of two diffeomorphisms, it must also be a diffeomorphism.