SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205C — Part 3

Fall 2003

III. Global theory of smooth manifolds and mappings

III.1 : Basic definitions and examples

Additional exercises

1. Let $\mathcal{E}$ be the smooth $\mathcal{C}^\infty$ atlas on $\mathbf{R}$ whose only chart is the identity, let $h : \mathbf{R} \to \mathbf{R}$ be the map defined by $h(x) = x^3$, and let $\mathcal{E}_h$ be the $\mathcal{C}^\infty$ atlas for $\mathbf{R}$ whose sole chart is $(\mathbf{R}, h)$. Prove that the map $h : (\mathbf{R}, \mathcal{E}) \to (\mathbf{R}, \mathcal{E}_h)$ is a diffeomorphism even though $h : (\mathbf{R}, \mathcal{E}) \to (\mathbf{R}, \mathcal{E})$ is a smooth map whose inverse is not smooth. — Generalize this result to an arbitrary continuous map $(U, \mathcal{E}) \to (U, \mathcal{E}_h)$ where h is open in $\mathbf{R}^n$ and h is a homeomorphism from U to itself.

SOLUTION.

By the weak criterion, to check the smoothness of a map f it is enough to find a covering collection of charts in the atlases having the form (U_α, h_α) and (V_β, k_β) such that $f(h(U_\alpha)) \subset V_\beta$ and "$k^{-1} f h$" is smooth. In this situation we only need one chart each for the domain and codomain; namely, $(\mathbf{R}, \mathrm{id}_\mathbf{R}$ for $\mathcal{E}$ and $(\mathbf{R}, h)$ for $\mathcal{E}_h$. In this case the local composite reduces to $h^{-1} \circ h = \mathrm{id}_\mathbf{R}$ Therefore the map h is smooth with respect to the given smooth structures. Consider now the inverse map $h^{-1} : (\mathbf{R}, \mathcal{E}_h) \to (\mathbf{R}, \mathcal{E})$; in this case one can proceed similarly and see that the local map given by the same charts is again $h^{-1} \circ h$.

More generally, if h is an arbitrary homeomorphism from and open subset $U \subset \mathbf{R}^n$ to itself, then $h : U \to U$ defines a diffeomorphism from $(U, \mathcal{E})$ to $(U, \mathcal{E}_h)$, regardless of what smoothness properties h or its inverse might possess.∎

2. Let $1 \le s < r \le \infty$, and let $(M, \mathcal{A})$ be a smooth $\mathcal{C}^r$ manifold where $\mathcal{A}$ is the maximal the notes we stated that $\mathcal{A}$ is also a $\mathcal{C}^s$ atlas but not a maximal $\mathcal{C}^s$-atlas. Prove the second part of this assertion. [*Hint:* There is a smooth $\mathcal{C}^s$ diffeomorphism of $\mathbf{R}^n$ that is not a smooth $\mathcal{C}^r$ diffeomorphism by results in the notes. Why is the analogous statement true if $\mathbf{R}^n$ is replaced by an open disk in $\mathbf{R}^n$? Use this to add extra charts to $\mathcal{A}$ such that the larger object is still a smooth $\mathcal{C}^s$ atlas.]

SOLUTION.

Follow the hint. The last result in Section II.1 constructs a diffeomorphism f from $\mathbf{R}$ to itself that is $\mathcal{C}^r$ but not $\mathcal{C}^{r+1}$. Since the inverse tangent map defines a $\mathcal{C}^\infty$ map from $\mathbf{R}$ to the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, the map $\arctan(\, h(\tan x)\,)$ defines a $\mathcal{C}^r$ diffeomorphism of $(-\frac{\pi}{2}, \frac{\pi}{2})$ to itself that is not $\mathcal{C}^{r+1}$. If g is the diffeomorphism so constructed, then we may construct a diffeomorphism g_0 of any open interval $(-\varepsilon, \varepsilon)$ to itself by taking $g_0 = T \circ g \circ T^{-1}$, where T is the unique linear homeomorphism from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $(-\varepsilon, \varepsilon)$. Taking repeated products of this map with the identity on the latter interval we obtain a $\mathcal{C}^r$ diffeomorphism from $(-\varepsilon, \varepsilon)^n$ to itself that is not $\mathcal{C}^{r+1}$. Similarly, if $\mathbf{x}$ is an arbitrary point in $\mathbf{R}^n$ with coordinates x_j, we have a $\mathcal{C}^r$ diffeomorphism from $\prod_j \, (x_j - \varepsilon, x_j + \varepsilon)^n$ to itself that is not $\mathcal{C}^{r+1}$.

Suppose now that we have a typical chart (U, h) in our maximal $\mathcal{C}^s$ atlas where $s > r$. If we restrict to any open subset $V \subset U$, then by maximality of the atlas we also know that $(V, h|V)$ is a smooth chart. Take V to be an open hypercubical region as in the previous paragraph, and consider the new chart

$$\left(V, (h|V) \circ g_0\right) .$$

This cannot lie in the maximal $\mathcal{C}^s$ atlas because the map "$(h|V)^{-1} \circ (h|V) \circ g_0$"$= g_0$ is not $\mathcal{C}^{r+1}$. However, we claim that if we add this new chart to the maximal $\mathcal{C}^{r+1}$ atlas we obtain a $\mathcal{C}^r$ atlas that properly contains the maximal $\mathcal{C}^{r+1}$ atlas. To keep the notation concise let us denote the new chart by (V, ℓ).

We need to show that if (W, k) is any other chart in the maximal $\mathcal{C}^{r+1}$ atlas then the composite "$k^{-1}\ell$" is smooth of class $\mathcal{C}^r$. But this follows because the latter map is given by the composite of the $\mathcal{C}^s$ map "$k^{-1}h$ and the $\mathcal{C}^r$ map g_0.∎

3. This exercise asks for a verification of a statement in the discussion of lens spaces. We recall the basic setting: Given a finite cyclic group $\mathbf{Z}_k$ of order k, and a positive integer n, let $(m_1, \cdots, m_n)$ be an ordered n-tuple of positive integers less than k such that each m_j is prime to k. Then a topological action of $\mathbf{Z}_k$ on $S^{2n-1} \subset \mathbf{C}^n \cong \mathbf{R}^{2n}$ is defined by the formula

$$g^j(z_1, \cdots, z_n \quad = \quad (\alpha^{m_1} z_1, \cdots, \alpha^{m_n} z_n)$$

where g denotes a standard generator of $\mathbf{Z}_k$ and $\alpha = \exp(2\pi i/k)$. Prove that this is a free action on $\mathbf{C}^n - \{\mathbf{0}\}$; *i.e.*, $g^j\mathbf{z} \neq \mathbf{z}$ if $j \not\equiv 0(k)$ and $\mathbf{z} \neq \mathbf{0}$.

SOLUTION.

This just reduces to a standard fact about vectors: If we are given a scalar c and a vector $\mathbf{v}$ then $c\mathbf{v} = \mathbf{v}$ if and only if $c = 1$ or $\mathbf{v} = \mathbf{0}$. In our case $c = g^j$ where $g = \exp(2\,m\pi\,i/k)$ and m is relatively prime to k, so that $g^j = 1$ if and only if $jm \equiv 0(k)$ which is equivalent to $j \equiv 0(k)$.∎

4. THE PROOF OF THIS EXERCISE WILL BE DEFERRED.

5. Let M and N be smooth manifolds, and let $h : M \to N$ be a continuous mapping. Prove that h is smooth if and only if for each open subset $V \subset N$ and each smooth function $f : V \to \mathbf{R}$ the composite "$f \circ h$": $h^{-1}(V) \to \mathbf{R}$ is smooth.

SOLUTION.

First of all, if V is an open subset of N then h defines a continuous mapping $h_V : h^{-1}(V) \to V$; we claim this map is smooth if h is smooth. This is true because $h|h^{-1}(V)$ is smooth and its image is contained in the open subset $V \subset N$. It follows that if h is smooth then $f \circ (h|h^{-1}(V))$ is smooth.

Conversely, suppose that h is continuous and for each smooth function $f : V \to \mathbf{R}$ the composite $f \circ (h|h^{-1}(V))$ is smooth. Suppose that V is an open set that is the image of a smooth coordinate chart (W, k). Denote the standard coordinate functions on W by z_j; it then follows that $w_j = z_j \circ k^{-1}$ is a smooth real valued function on V, and by hypothesis the latter in turn implies that the composite $w_j \circ h_V$ is smooth for all j. However, this means that the coordinates of the vector valued function $k^{-1} \circ h_V$ are all smooth, which means that $k^{-1} \circ (h|h^{-1}(V))$ itself must be smooth.

If we now take an open covering of N by images V_α of coordinate charts, the reasoning of the preceding paragraphs shows that the restrictions of h to each of the sets $h^{-1}(V_\alpha)$ are smooth. Since these inverse images form an open covering, it follows that h itself is smooth.∎

6. **DISREGARD THIS PROBLEM.**

III.2 : Constructions on smooth manifolds

(Conlon, §§1.7, 3.7)

Additional exercises

1. (*i*) Let X, Y and Z be smooth manifolds and let $\times$ denote the usual cartesian product. Prove that

$$X \times (Y \times Z)$$

is a direct product of smooth manifolds as defined in the notes.

SOLUTION.

This is a special case of Exercise 3.■

(*ii*) Let A, B, C and D be smooth manifolds and let $\times$ denote the usual cartesian product. Prove that

$$(A \times B) \ \times \ (C \times D)$$

is a direct product of smooth manifolds as defined in the notes.

[*Note:* These may all be viewed as special cases of a more general result.]

SOLUTION.

This is also a special case of Exercise 3.■

2. Let X and Y be smooth manifolds and let $\tau : X \times Y \to Y \times X$ be the "twist map" which sends (x, y) to (y, x) for all x and y. Prove that τ is a diffeomorphism. [*Hint:* Consider the analogous map $\tau' : Y \times X \to X \times Y$.]

SOLUTION.

The proof is nearly the same as the proof of the corresponding result in Section I.4, the main difference being thatone must substitute "smooth manifold," "smooth mapping" and "diffeomorphism" for "topological space," "continuous function" and "homeomorphism" throughout the discussion.■

(*ii*) Let X be a smooth manifold and let $T : X \times X \times X \to X \times X \times X$ be the map that cyclically permutes the coordinates: $T(x, y, z) = (z, x, y)$. Prove that T is a diffeomorphism. [*Hint:* What is the test for smoothness of a map into a product? Can you write down an explicit formula for the inverse function?]

SOLUTION.

The proof is nearly the same as the proof of the corresponding result in Section I.4, the main difference being thatone must substitute "smooth manifold," "smooth mapping" and "diffeomorphism" for "topological space," "continuous function" and "homeomorphism" throughout the discussion.■

3. ("A product of products is a product.") Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a finite family of smooth manifolds, and let $\mathcal{A} = \cup\{\mathcal{A}_\beta \mid \beta \in \mathcal{B}\}$ be a partition of $\mathcal{A}$. Construct a diffeomorphism from $\prod\{A_\alpha \mid \alpha \in \mathcal{A}\}$ to the set

$$\prod_\beta \{\prod\{A_\alpha \mid \alpha \in \mathcal{A}_\beta\} \ \} \ .$$

SOLUTION.

The proof is nearly the same as the proof of the corresponding result in Section I.4, the main difference being thatone must substitute "smooth manifold," "smooth mapping" and "diffeomorphism" for "topological space," "continuous function" and "homeomorphism" throughout the discussion.■

4. Let $\mathcal{A}$ be some nonempty set, let $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ and $\{Y_\alpha \mid \alpha \in \mathcal{A}\}$ be finite families of smooth manifolds, and for each $\alpha \in \mathcal{A}$ suppose that $f_\alpha : X_\alpha \to Y_\alpha$ is a diffeomorphism. Prove that the product map

$$\prod_\alpha f_\alpha \ : \ \prod_\alpha X_\alpha \longrightarrow \prod_\alpha Y_\alpha$$

is also a diffeomorphism. [*Hint:* What happens when you take the product of the inverse maps?]

SOLUTION.

The proof is nearly the same as the proof of the corresponding result in Section I.4, the main difference being thatone must substitute "smooth manifold," "smooth mapping" and "diffeomorphism" for "topological space," "continuous function" and "homeomorphism" throughout the discussion.■

5. Prove that $\mathbf{R}^n - \{0\}$ is diffeomorphic to $S^{n-1} \times \mathbf{R}$.

SOLUTION.

Let π_1 and π_2 denote the projections from $S^{n-1} \times \mathbf{R}$ onto S^{n-1} and $\mathbf{R}$ respectively. Define a smooth map f from $\mathbf{R}^n - \{0\}$ to $S^{n-1} \times \mathbf{R}$ such that $\pi_1 \circ f(\mathbf{v}) = |\mathbf{v}|^{-1} \cdot \mathbf{v}$ and $\pi_2 \circ f(\mathbf{v}) = \log|\mathbf{v}|$. Similarly, define a smooth map g in the opposite direction by $g(\mathbf{w}, t) = e^t \cdot \mathbf{w}$. Direct computation then show that $f \circ g(\mathbf{w}, t) = (\mathbf{w}, t)$ and $g \circ f(\mathbf{v}) = \mathbf{v}$, so that f and g are inverse to each other.■

III.3 : Smooth approximations

(Conlon, §§3.5, 3.8)

Problems from Conlon, pp. 116 – 117

3.8.3.

SOLUTION.

This is completed in Subsection III.3.3 of the lecture notes.■

3.8.5.

SOLUTION.

Let $\text{Diff}_c(M)$ be the group of diffeomorphisms that are the identity off some compact set. We need to show that this is a subgroup of the full group of diffeomorphisms.

Following Conlon, the support of a diffeomorphism f is the closure of the set of points where $f(x) \neq x$. We shall call this set $\text{Supp}(f)$.

First of all $\text{Diff}_c(M)$ contains id_M because $\text{Supp}(\text{id}_M) = \emptyset$.

Next, suppose that $f, g \in \mathrm{Diff}_c(M)$. If $x \notin \mathrm{Supp}(f) \cup \mathrm{Supp}(g)$ then $f(x) = g(x) = x$ so that $g \circ f(x) = x$. Therefore we have

$$\mathrm{Supp}(g \circ f) \quad \subset \mathrm{Supp}(f) \cup \mathrm{Supp}(g)$$

and since the right hand side is a union of two compact sets it follows that the left hand side is compact. Therefore we have $g \circ f \in \mathrm{Diff}_c(M)$.

Finally, if $f \in \mathrm{Diff}_c(M)$, then $f(x) \neq x \Longleftrightarrow x \neq f^{-1}(x)$, so that

$$\mathrm{Supp}(f) \quad = \quad \mathrm{Supp}(f^{-1})$$

and consequently $f^{-1} \in \mathrm{Diff}_c(M)$.∎

3.8.6.

SOLUTION.

We need to show that compactly supported isotopy is an equivalence relation on the group in the previous problem. By results in Section III.3 we might as well assume that these smooth isotopies are strongly admissible.

A diffeomorphism $f \in \mathrm{Diff}_c(M)$, is compactly supported isotopic to itself because the suppport of the trivial isotopy $H : M \times [0,1] \to M$ defined by $H(x,t) = f(x)$ has the same compact support as f.

Suppose that f is compactly supported isotopic to g and H is a compactly supported isotopy. If $H^* : M \times [0,1] \to M$ is defined by

$$H^*(x,t) \quad = \quad H(x, 1-t)$$

then H^* is a smooth compactly supported isotopy from g to f and the support of H^* is equal to the (compact) support of H.

Suppose that f is compactly supported isotopic to g with compactly supported isotopy H and g is compactly supported isotopic to h with compactly supported isotopy K. Then one has a smooth isotopy L from f to h defined by H on $M \times [0, \frac{1}{2}]$ and by K on $M \times [\frac{1}{2}, 1]$, and the support of this isotopy is the union of the (compact) supports if H and K. — Combining these, we see that compactly supported isotopy is an equivalence relation.∎

Further conclusions. The equivalence class of the identity is a normal subgroup and the other equivalence classes are cosets of that subgroup. Details are left to the reader (one needs to show that the equivalence class of the identity is closed under multiplication, taking inverses, and conjugation by an arbitrary diffeomorphism with compact support — in fact, it is closed under conjugation with respect to an arbitrary diffeomorphism, for the support of $h^{-1} \circ f \circ h$ is the image of the support of f under h^{-1}).∎

Additional exercises

1. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a diffeomorphism.

(i) Why is the derivative f' always positive or always negative?

SOLUTION.

The derivative is always nonzero because $g'(f(x)) \cdot f'(x) = 1$ by the Chain Rule. If it were both positive and negative then by the Intermediate Value Property it would be zero somewhere. In our situation smoothness implies that the derivative is continuous so we can use the usual Intermediate Value Theorem for continuous functions, but everything goes through for any differentiable functions because derivatives always have the Intermediate Value Property.■

(*ii*) Prove that f is smoothly isotopic to the identity if f' is always positive and smoothly isotopic to minus the identity if f' is always negative. [*Hints:* It will simplify things to note first that one can find a diffeomorphism isotopic to f such that $f(0) = 0$. If $f' > 0$, what can one say about the straight line homotopy from f to the identity?]

SOLUTION.

Let $\psi : \mathbf{R} \to \mathbf{R}$ be a smooth map taking values in $[0, 1]$ such that ψ is nondecreasing and there is some $\varepsilon > 0$ such that $\psi = 0$ for $t < \varepsilon$ and $\psi = 1$ for$t > 1 - \varepsilon$. Define a strongly admissible smooth isotopy A such that $A(x, t) = f(x) + (1 - \psi(t))f(0)$; then the diffeomorphism A_0 given by $A||bfR \times \{0\}$ is equal to f and the diffeomorphism A_1 given by $A|\mathbf{R} \times \{1\}$ satisfies $A_1(0) = 0$. This verifies the first point in the hint. Let $g = A_1$; it will suffice to show that g satisfies the conditions in the exercise. shall

Now define another homotopy by

$$B(x,t) \quad = \quad (1 - \psi(t))\, g(x) \ + \ \psi(t) \cdot \sigma x$$

where σ is the sign of $g'(x)$, which we know is the same for all values of x. It follows immediately that the first partial derivative of this function is always nonzero, and in fact the sign of the first partial derivative is equal to the sign σ; this merely reflects the fact that if u and v are nonzero real numbers with the same sign, then the closed interval with endpoints u and v does not contain the origin. This implies that each map $B|\mathbf{R} \times \{t\}$ is strictly increasing and therefore is 1–1. To see that each B_t is onto, by the Intermediate Value Theorem it suffices to show that

$$\lim_{x \to \pm\infty} B_t(x) \quad = \quad \pm\,\sigma\infty \ .$$

Since g is a diffeomorphism we know this holds for $t = 0$. Let $M > 0$ and assume that $\sigma = +1$. Then there is a $K > 0$ such that $x > K$ or $x < -K$ implies $g(x) > M$ or $g(x) < -M$. Therefore if L is the larger of K and M, then $x > L$ or $x < -L$ implies $B_t(x) > M$ or $B_t(x) < M$. Therefore in this case we have shown that g is smoothly strongly admissibly isotopic to the identity. We could dispose of the case where $\sigma = -1$ similarly, but it is faster to let $h = \sigma \cdot g$ and use the case already established to prove that h is smoothly strongly admissibly isotopic to the identity. If B is the relevant isotopy, then $\sigma \cdot B$ will be a smoothly strongly admissible isotopy from g to σ times the identity.■

(*iii*) Prove that every diffeomorphism of S^1 to itself is smoothly isotopic to either the identity or complex conjugation.

SOLUTION.

Given a diffeomorphism f of S^1 to itself, we can lift it to a smooth map F from $\mathbf{R}$ to itself. In fact, given a real number t_0 such that $\exp(2\pi\, i\, t_0) = f(1)$ we can find a smooth lifting such that $F(0) = t_0$. Similarly, if g is the inverse to f we can find a unique lifting G such that $G(t_0) = 0$, and in fact F and G are inverses to each other. Note that both F and G satisfy the basic condition $H(t + n) = H(t) + \Delta \cdot n$ for all integers n, where Δ is some integer.

We claim that $\Delta = \pm 1$; perhaps the fastest proof of this is that Δ corresponds to the degree of f or g and the degree of a homeomorphism is always $\pm\, 1$. Of course the sign is closely related to the sign of the derivative of F. If C is one of the isotopies constructed as in the preceding part of the exercise for our special choice of F, then it follows immediately that $C(x, t+n) = C(x,t) + \Delta \cdot n$. Therefore C passes to a continuous map D from $S^1 \times [0,1]$ to S^1, and by construction this map on the quotients is a strongly admissible smooth isotopy if each map D_t is 1–1 and onto. One quick way of checking this is to note that C_t must map an interval $(a, a+1)$ to another interval $(b, b+1)$ in a 1–1 onto fashion, for the latter implies that D_t must be 1–1 onto.

2. Let M be a smooth manifold. Two diffeomorphisms f and g from M to itself are said to be smoothly *concordant* or *pseudo-isotopic* if there is a homeomorphism H from $M \times [0,1]$ to itself with the following properties:

(1) The homeomorphism sends $M \times \{0\}$ to itself by f and $M \times \{1\}$ to itself by g.

(2) The homeomorphism is a diffeomorphism on $M \times (0,1)$.

(3) For each $x \in M$ there is an open neighborhood U and an $\varepsilon > 0$ such that the restrictions of H to $U \times [0, \varepsilon)$ and $U \times (\varepsilon, 1]$ depend only on the first variable. (If M is compact this is equivalent to saying that H is given by f on some open set of the form $M \times [0, \delta)$ and by g on some open set of the form $M \times (1-\delta, 1]$.)

Prove that concordance defines an equivalence relation on diffeomorphisms of M and that isotopic diffeomorphisms are concordant. [The difference is that a concordance does not send the level submanifolds $M \times \{t\}$ into themselves. Determining the relation between concordance and isotopy is a deep and difficult question that was essentially answered in the nineteen seventies by A. Hatcher and J. Wagoner for manifolds of sufficiently large dimension.]

SOLUTION.

The proof of this proceeds very much like the arguments for smooth homotopies.∎

III.4 : Amalgamation theorems

(Conlon, §1.3)

Additional exercises

1. Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of topological spaces, and let $X = \coprod_\alpha A_\alpha$. Prove that X is locally connected if and only if each A_α is locally connected.

SOLUTION.

($\Longrightarrow$) If X is locally connected then so is every open subset. But each A_α is an open subset, so each is locally connected.∎

($\Longleftarrow$) We need to show that for each $x \in X$ and each open set U containing x there is an open subset $V \subset U$ such that $x \in V$ and V is connected. There is a unique α such that $x = i_\alpha(a)$ for some $a \in A_\alpha$. Let $U_0 = i_\alpha^{-1}(U)$. Then by the local connectedness of A_α and the openness of U_0 there is an open connected set V_0 such that $x \in V_0 \subset U_0$. If $V = i_\alpha(V_0)$, then V has the required properties.∎

2. In the preceding exercise, formulate and prove necessary and sufficient conditions on $\mathcal{A}$ and the sets A_α for the space X to be compact.

SOLUTION.

X is compact if and only if each A_α is compact and there are only finitely many (nonempty) subsets in the collection.

The ($\Longrightarrow$) implication follows because each A_α is an open and closed subspace of the compact space X and hence compact, and the only way that the open covering $\{ A_\alpha \}$ of X, which consists of pairwise disjoint subsets, can have a finite subcovering is if it contains only finitely many subsets. To prove the reverse implication, one need only use a previous exercise which shows that a finite union of compact subspaces is compact.■

3. Prove that $\mathbf{RP}^2$ can be constructed by identifying the edge of a Möbius strip with the edge circle on a closed 2-dimensional disk by filling in the details of the following argument: Let $A \subset S^2$ be the set of all points $(x, y, z) \in S^2$ such that $|z| \leq \frac{1}{2}$, and let B be the set of all points where $|z| \geq \frac{1}{2}$. If $T(x) = -x$, then $T(A) = A$ and $T(B) = B$ so that each of A and B (as well as their intersection) can be viewed as a union of equivalence classes for the equivalence relation that produces $\mathbf{RP}^2$. By construction B is a disjoint union of two pieces $B_\pm$ consisting of all points where sign$(z) = \pm 1$, and thus it follows that the image of B in the quotient space is homeomorphic to $B_+ \cong D^2$. Now consider A. There is a homeomorphism h from $S^1 \times [-1, 1]$ to A sending (x, y, t) to $(\alpha(t)x,\ \alpha(t)y,\ \frac{1}{2}t)$ where

$$\alpha(t) \quad = \quad \sqrt{1 - \frac{t^2}{4}}$$

and by construction $h(-v) = -h(v)$. The image of A in the quotient space is thus the quotient of $S^1 \times [-1, 1]$ modulo the equivalence relation $u \sim v \iff u = \pm v$. This quotient space is in turn homeomorphic to the quotient space of the upper semicircular arc S^1_+ (all points with nonnegative y-coordinate) modulo the equivalence relation generated by setting $(-1, 0, t)$ equivalent to $(1, 0, -t)$, which yields the Möbius strip. The intersection of this subset in the quotient with the image of B is just the image of the closed curve on the edge of B_+, which also represents the edge curve on the Möbius strip.

FURTHER DETAILS.

We shall fill in some of the reasons that were left unstated in the sketch given above.

Let $A \subset S^2$ be the set of all points $(x, y, z) \in S^2$ such that $|z| \leq \frac{1}{2}$, and let B be the set of all points where $|z| \geq \frac{1}{2}$. If $T(x) = -x$, then $T(A) = A$ and $T(B) = B$ [etc.]

This is true because if $T(v) = w$, then the third coordinates of both points have the same absolute values and of course they satisfy the same inequality relation with respect to $\frac{1}{2}$.

By construction B is a disjoint union of two pieces $B_\pm$ consisting of all points where sign$(z) = \pm 1$,

This is true the third coordinates of all points in B are nonzero.

There is a homeomorphism h from $S^1 \times [-1, 1]$ to A sending (x, y, t) to $(\alpha(t)x, \alpha(t)y, \frac{1}{2}t)$ where

$$\alpha(t)s \quad = \quad \sqrt{1 - \frac{t^2}{4}}$$

One needs to verify that h is 1–1 onto; this is essentially an exercise in algebra. Since we are dealing with compact Hausdorff spaces, continuous mappings that are 1–1 onto are automatically homeomorphisms.

This quotient space $[S^1 \times [-1,1]$ *modulo the equivalence relation* $u \sim v \iff u = \pm v]$ *is in turn homeomorphic to the quotient space of the upper semicircular arc* S^1_+ *(all points with nonnegative* y*-coordinate) modulo the equivalence relation generated by setting* $(-1,0,t)$ *equivalent to* $(1,0,-t)$*, which yields the Möbius strip.*

Let $\mathcal{A}$ and $\mathcal{B}$ be the respective equivalence relations on $S^1_+ \times [-1,1]$ and $S^1 \times [-1,1]$, and let $\mathbf{A}$ and $\mathbf{B}$ be the respective quotient spaces. By construction the inclusion $S^1_+ \times [-1,1] \subset S^1 \times [-1,1]$ passes to a continuous map of quotients, and it is necessary and sufficient to check that this map is 1–1 and onto. This is similar to a previous exercise. Points in $S^1 - S^1_+$ all have negative second coordinates and are equivalent to unique points with positive second coordinates. This implies that the mapping from $\mathbf{A}$ to $\mathbf{B}$ is 1–1 and onto at all points except perhaps those whose second coordinates are zero. For such points the equivalence relations given by $\mathcal{A}$ and $\mathcal{B}$ are identical, and therefore the mapping from $\mathbf{A}$ to $\mathbf{B}$ is also 1–1 and onto at all remaining points.∎

4. Suppose that the topological space X is a union of two closed subspaces A and B, let $C = A \cap B$, let $h : C \to C$ be a homeomorphism, and let $A \cup_h B$ be the space formed from $A \sqcup B$ by identifying $x \in C \subset A$ with $h(x) \in C \subset B$. Prove that $A \cup_h B$ is homeomorphic to X if h extends to a homeomorphism $H : A \to A$, and give an example for which X is not homeomorphic to $A \cup_h B$. [*Hint:* Construct the homeomorphism using H in the first case, and consider also the case where $X = S^1 \sqcup S^1$, with $A_\pm == S^1_\pm \sqcup S^1_\pm$; then $C = \{\pm 1\} \times \{1,2\}$, and there is a homeomorphism from h to itself such that $A_+ \cup_h A_-$ is connected.]

SOLUTION.

We can and shall view X as $A \cup_{\mathrm{id}} B$.

Consider the map $F_0 : A \sqcup B \to A \sqcup B$ defined by H^{-1} on A and the identity on B. We claim that this passes to a unique continuous map of quotients from X to $A \cup_h B$; *i.e.*, the map F_0 sends each nonatomic equivalence classes $\{ (c,1), (c,2) \}$ for $X = A \cup_{\mathrm{id}} B$ to a nonatomic equivalence class of the form $\{ (u,1), (h(u),2) \}$ for $A \cup_h B$. Since F_0 sends $(c,1)$ to $(h^{-1}(c),1)$ and $(c,2)$ to itself, we can verify the compatibility of F_0 with the equivalence relations by taking $u = h^{-1}(c)$. Passage to the quotients then yields the desired map $F : X \to A \cup_h B$.

To show this map is a homeomorphism, it suffices to define Specifically, start with $G_0 = F_0^{-1}$, so that $G_0 = H$ on A and the identity on B. In this case it is necessary to show that a nonatomic equivalence class of the form $\{ (u,1), (h(u),2) \}$ for $A \cup_h B$ gets sent to a nonatomic equivalence class of the form $\{ (c,1), (c,2) \}$ for $X = A \cup_{\mathrm{id}} B$. Since G_0 maps the first set to $\{ (h(u),1), (h(u),2) \}$ this is indeed the case, and therefore G_0 also passes to a map of quotients which we shall call G.

Finally we need to verify that F and G are inverses to each other. By construction the maps F_0 and G_0 satisfy $F([y]) = [F_0(y)]$ and $G([z]) = [G_0(z)]$, where square brackets denote equivalence classes. Therefore we have

$$G \circ F([y]) = G\left([F_0(y)]\right) = [G_0\left(F_0(y)\right)]$$

which is equal to $[y]$ because F_0 and G_0 are inverse to each other. Therefore $G \circ F$ is the identity on X. A similar argument shows that $F \circ G$ is the identity on $A \cup_h B$.

To construct the example where X is **not** homeomorphic to $A \cup_h B$, we follow the hint and try to find a homeomorphism of the four point space $\{\pm 1\} \times \{1,2\}$ to itself such that X is **not** homeomorphic to $A \cup_h B$ is connected; this suffices because we know that X is not connected. Sketches on paper or physical experimentation with wires or string are helpful in finding the right formula.

Specifically, the homeomorphism we want is given as follows:

$$\begin{array}{rcl} (-1,1)\in A_+ & \longrightarrow & (1,2)\in A_- \\ (1,1)\in A_+ & \longrightarrow & (1,1)\in A_- \\ (1,2)\in A_+ & \longrightarrow & (-1,1)\in A_- \\ (-1,2)\in A_+ & \longrightarrow & (-1,2)\in A_- \end{array}$$

The first of these implies that the images of $S^1_+ \times \{2\}$ and $S^1_- \times \{1\}$ lie in the same component of the quotient space, the second of these implies that the images of $S^1_- \times \{1\}$ and $S^1_+ \times \{1\}$ both lie in the same component, and the third of these implies that the images of $S^1_+ \times \{2\}$ and $S^1_- \times \{2\}$ also lie in the same component. Since the entire space is the union of the images of the connected subsets $S^1_\pm \times \{1\}$ and $S^1_\pm \times \{2\}$ it follows that $A \cup_h B$ is connected.■

FOOTNOTE.

The argument in the first part of the exercise remains valid if A and B are open rather than closed subsets.■

5. *One-point unions.* One conceptual problem with the disjoint union of topological spaces is that it is never connected except for the trivial case of one summand. In many geometrical and topological contexts it is extremely useful to construct a modified version of disjoint unions that is connected if all the pieces are. Usually some additional structure is needed in order to make such constructions.

In this exercise we shall describe such a construction for objects known as *pointed spaces* that are indispensable for many purposes (*e.g.*, the definition of fundamental groups as in Munkres). A pointed space is a pair (X, x) consisting of a topological space X and a point $x \in X$; we often call x the *base point*, and unless stated otherwise *the one point subset consisting of the base point is assumed to be closed*. If (Y, y) is another pointed space and $f : X \to Y$ is continuous, we shall say that *f is a base point preserving continuous map from (X, x) to (Y, y)* if $f(x) = y$, In this case we shall often write $f : (X, x) \to (Y, y)$. Identity maps are base point preserving, and composites of base point preserving maps are also base point preserving.

Given a finite collection of pointed spaces (X_i, x_i), define an equivalence relation on $\coprod_i X_i$ whose equivalence classes consist of $\coprod_j \{x_j\}$ and all one point sets y such that $y \notin \coprod_j \{x_j\}$. Define the *one point union* or *wedge*

$$\bigvee_{i=1}^{n} (X_j, x_j) \quad = \quad (X_1, x_1) \vee \cdots \vee (X_n, x_n)$$

to be the quotient space of this equivalence relation with the quotient topology. The base point of this space is taken to be the class of $\coprod_j \{x_j\}$.

(a) Prove that the wedge is a union of closed subspaces Y_j such that each Y_j is homeomorphic to X_j and if $j \neq k$ then $Y_j \cap Y_k$ is the base point. Explain why $\vee_k (X_k, x_k)$ is Hausdorff if and only if each X_j is Hausdorff, why $\vee_k (X_k, x_k)$ is compact if and only if each X_j is compact, and why $\vee_k (X_k, x_k)$ is connected if and only if each X_j is connected (and the same holds for arcwise connectedness).

SOLUTION.

For each j let $\mathbf{in}_j : X_j \to \coprod_k X_k$ be the standard injection into the disjoint union, and let

$$P : \coprod_k X_k \longrightarrow \bigvee_k (X_k, x_k)$$

be the quotient map defining the wedge. Define Y_j to be $P \circ \mathbf{in}_j(X_j)$. By construction the map $P \circ \mathbf{in}_j$ is continuous and 1–1; we claim it also sends closed subsets of X_j to closed subsets of the wedge. Suppose that $F \subset X_j$ is closed; then $P \circ \mathbf{in}_j(F)$ is closed in the wedge if and only if its inverse image under P is closed. But this inverse image is the union of the closed subsets $\mathbf{in}_j(F)$ and $\coprod_k \{x_k\}$ (which is a finite union of one point subsets that are assumed to be closed). It follows that Y_j is homeomorphic to $X + j$. The condition on $Y_k \cap Y_\ell$ for $k \neq \ell$ is an immediate consequence of the construction.

The assertion that the wedge is Hausdorff if and only if each summand is follows because a subspace of a Hausdorff space is Hausdorff, and a finite union of closed Hausdorff subspaces is always Hausdorff (by a previous exercise).

To verify the assertions about compactness, note first that for each j there is a continuous collapsing map q_j from $\vee_k\ (X_k, x_k)$ to (X_j, x_j), defined by the identity on the image of (X_j, x_j) and by sending everything to the base point on every other summand. If the whole wedge is compact, then its continuous under q_j, which is the image of X_j, must also be compact. Conversely if the sets X_j are compact for all j, then the (finite!) union of their images, which is the entire wedge, must be compact.

To verify the assertions about connectedness, note first that for each j there is a continuous collapsing map q_j from $\vee_k\ (X_k, x_k)$ to (X_j, x_j), defined by the identity on the image of (X_j, x_j) and by sending everything to the base point on every other summand. If the whole wedge is connected, then its continuous under q_j, which is the image of X_j, must also be connected. Conversely if the sets X_j are connected for all j, then the union of their images, which is the entire wedge, must be connected because all these images contain the base point. Similar statements hold for arcwise connectedness and follow by inserting "arcwise" in front of "connected" at every step of the argument.■

(b) Let $\varphi_j : (X_j, x_j) \to \vee_k\ (X_k, x_k)$ be the composite of the injection $X_j \to \coprod_k\ X_k$ with the quotient projection; by construction φ_j is base point preserving. Suppose that (Y, y) is some arbitrary pointed space and we are given a sequence of base point preserving continuous maps $F_j : (X_j, x_j) \to (Y, y)$. Prove that there is a unique base point preserving continuous mapping

$$F : \vee_k\ (X_k, x_k) \to (Y, y)$$

such that $F \circ \varphi_j = F_j$ for all j.

SOLUTION.

To prove existence, first observe that there is a unique continuous map $\widetilde{F} : \coprod_k X_k \to Y$ such that $\mathbf{in}_j \circ \widetilde{F} = F_j$ for all j. This passes to a unique continuous map F on the quotient space $\vee_k\ (X_k, x_k)$ because $\widetilde{F}$ is constant on the equivalence classes associated to the quotient projection P. This constructs the map we want; uniqueness follows because the conditions prescribe the definition at every point of the wedge.■

(c) In the infinite case one can carry out the set-theoretic construction as above but some care is needed in defining the topology. Show that if each X_j is Hausdorff and one takes the so-called *weak topology* whose closed subsets are generated by the family of subsets $\varphi_j(F)$ where F is closed in X_j for some j, then [1] a function h from the wedge into some other space Y is continuous if and only if each composite $h \circ \varphi_j$ is continuous, [2] the existence and uniqueness theorem for mappings from the wedge (in the previous portion of the exercise) generalizes to infinite wedges with the so-called weak topologies.

SOLUTION.

Strictly speaking, one should verify that the so-called weak topology is indeed a topology on the wedge. We shall leave this to the reader.

To prove [1], note that ($\Longrightarrow$) is trivial. For the reverse direction, we need to show that if E is closed in Y then $h^{-1}(E)$ is closed with respect to the so-called weak topology we have defined. The subset in question is closed with respect to this topology if and only if $h^{-1}(E) \cap \varphi(X_j)$ is closed in $\varphi(X_j)$ for all j, and since φ_j maps its domain homeomorphically onto its image, the latter is true if and only if $\varphi^{-1} \circ h^{-1}(E)$ is closed in X_j for all j. But these conditions hold because each of the maps $\varphi_j \circ h$ is continuous. To prove [2], note first that there is a unique set-theoretic map, and then use [1] to conclude that it is continuous.■

(d) Suppose that we are given an infinite wedge such that each summand is Hausdorff and contains at least two points. Prove that the wedge with the so-called weak topology is not compact.

SOLUTION.

For each j let $y_j \in X_j$ be a point other than x_j, and consider the set E of all points y_j. This is a closed subset of the wedge because its intersection with each set $\varphi(X_j)$ is a one point subset and hence closed. In fact, every subset of E is also closed by a similar argument (the intersections with the summands are either empty or contain only one point), so E is a discrete closed subset of the wedge. Compact spaces do not have infinite discrete closed subspaces, and therefore it follows that the infinite wedge with the weak topology is not compact.■

Remark. If each of the summands in **(d)** is compact Hausdorff, then there is a natural candidate for a *strong topology* on a countably infinite wedge which makes the latter into a compact Hausdorff space. In some cases this topology can be viewed more geometrically; for example, if each (X_j, x_j) is equal to $(S^1, 1)$ and there are countably infinitely many of them, then the space one obtains is the *Hawaiian earring* in $\mathbf{R}^2$ given by the union of the circles defined by the equations

$$\left(x - \frac{1}{2^k}\right)^2 \; + \; + \; y^2 \quad = \quad \frac{1}{2^{2k}} \; .$$

As usual, drawing a picture may be helpful. The $k^{\rm th}$ circle has center $(1/2^k,\, 0)$ and passes through the origin; the y-axis is the tangent line to each circle at the origin.

SKETCHES OF VERIFICATIONS OF ASSERTIONS.

If we are given an infinite sequence of compact Hausdorff pointed spaces $\{\,(X_n, x_n)\,\}$ we can put a compact Hausdorff topology on their wedge as follows. Let W_k be the wedge of the first k spaces; then for each k there is a continuous map

$$q_k : \bigvee_n (X_n, x_n) \quad \longrightarrow \quad W_k$$

(with the so-called weak topology on the wedge) that is the identity on the first k summands and collapses the remaining ones to the base point. These maps are in turn define a continuous function

$$\mathbf{q} : \bigvee_n (X_n, x_n) \quad \longrightarrow \quad \prod_k W_k$$

whose projection onto W_k is q_k. This mapping is continuous and 1–1; if its image is closed in the (compact!) product topology, then this defines a compact Hausdorff topology on the infinite wedge $\vee_n (X_n, x_n)$.

Here is one way of verifying that the image is closed. For each k let $c_k : W_k \to W_{k-1}$ be the map that is the identity on the first $(k-1)$ summands and collapses the last one to a point. Then we may define a continuous map C on $\prod_{k\geq 1} W_k$ by first projecting onto the product $\prod_{k\geq 2} W_k$ (forget the first factor) and then forming the map $\prod_{k\geq 2} W_k$. The image of $\mathbf{q}$ turns out to be the set of all points $\mathbf{x}$ in the product such that $C(\mathbf{x}) = \mathbf{x}$. Since the product is Hausdorff the image set is closed in the product and thus compact.

A comment about the compactness of the Hawaiian earring E might be useful. Let F_k be the union of the circles of radius 2^{-j} that are contained in E, where $j \leq k$, together with the closed disk bounded by the circle of radius $2^{-(k+1)}$ in E. Then F_k is certainly closed and compact. Since E is the intersection of all the sets F_k it follows that E is also closed and compact.∎

6. Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of topological spaces, and let $X = \coprod_\alpha A_\alpha$. Formulate and prove necessary and sufficient conditions on $\mathcal{A}$ and the sets A_α for the space X to be second countable, separable or Lindelöf.

SOLUTION.

For each property $\mathcal{P}$ given in the exercise, the space X has property $\mathcal{P}$ if and only if each A_α does and there are only finitely many α for which A_α is nonempty. The verifications for the separate cases are different and will be given in reverse sequence.

The Lindelöf property.

The proof in this case is the same as the proof we gave for compactness in an earlier exercise with "countable" replacing "finite" throughout.∎

Separability.

$(\Longrightarrow)$ Let D be the countable dense subset. Each A_α must contain some point of D, and by construction this point is not contained in any of the remaining sets A_β. Thus we have a 1–1 function from $\mathcal{A}$ to D sending α to a point $d(\alpha) \in A_\alpha \cap D$. This implies that the cardinality of $\mathcal{A}$ is at most $|D| \leq \aleph_0$.∎

$(\Longleftarrow)$ If D_α is a dense subset of A_α and $\mathcal{A}$ is countable, then $\cup_\alpha D_\alpha$ is a countable dense subset of $\mathcal{A}$.∎

Second countability.

$(\Longrightarrow)$ Since a subspace of a second countable space is second countable, each A_α must be second countable. Since the latter condition implies both separability and the Lindelöf property, the preceding arguments show that only countably many summands can be nontrivial.

$(\Longleftarrow)$ If $\mathcal{A}$ is countable and $\mathcal{B}_\alpha$ is a countable base for A_α then $\cup_\alpha \mathcal{B}_\alpha$ determines a countable base for X (work out the details!).∎