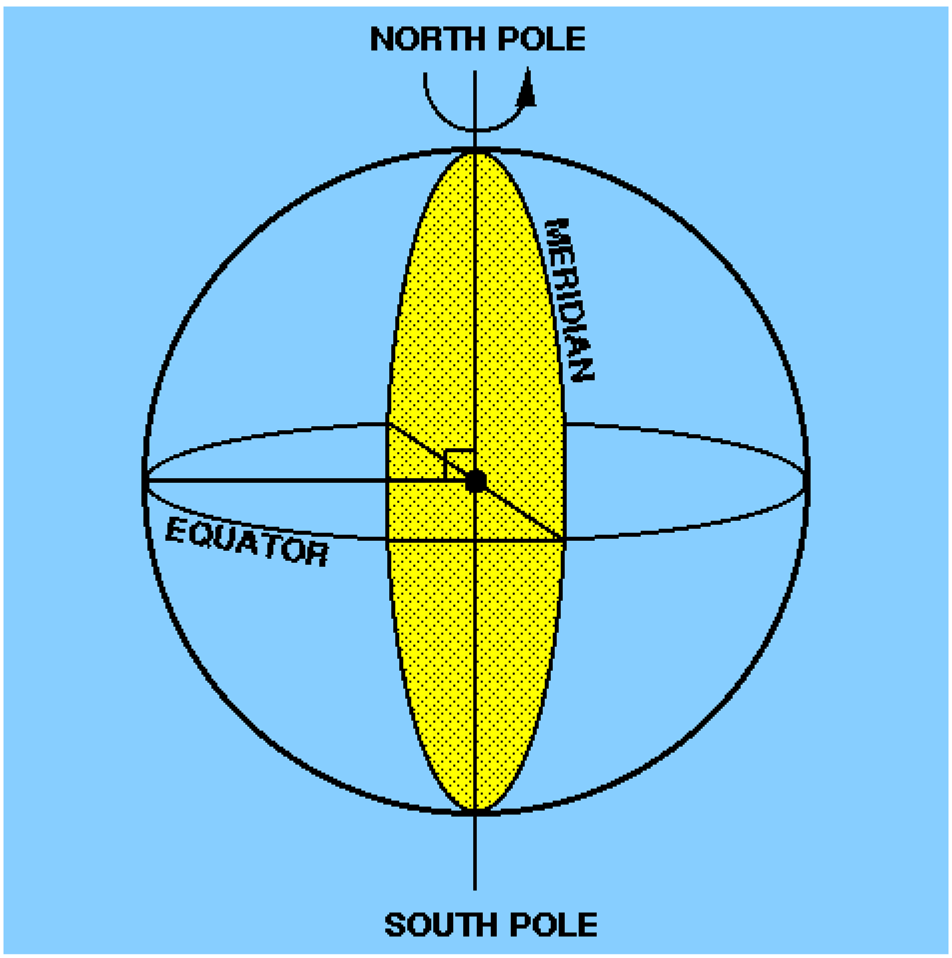
A submersion from the plane onto the 2 – sphere

The objective is to construct a smooth map **X** from the coordinate plane $\mathbb{R}^2$ onto the 2 – sphere $\mathbf{S}^2$ which is a ***regular surface parametrization*** at all points of $\mathbb{R}^2$; in other words, the derivative map $\boldsymbol{DX}$ from $\mathbb{R}^2$ to the space of $\mathbf{2 \times 3}$ matrices over $\mathbb{R}$ has rank **2** at all points of $\mathbb{R}^2$, or equivalently the cross product of the partial derivatives $\mathbf{X_1 \times X_2}$ is everywhere nonzero. We shall concentrate on describing the construction process itself; filling in the details can be done using methods and results from a graduate level course on smooth manifolds like Mathematics 205C.

We shall begin the construction with a closed curve on the sphere which goes around the equator once and then goes around the basic meridian pair (0 and 180 degrees) once. The initial and final point of this closed curve is located on the equator at 0 degrees longitude.



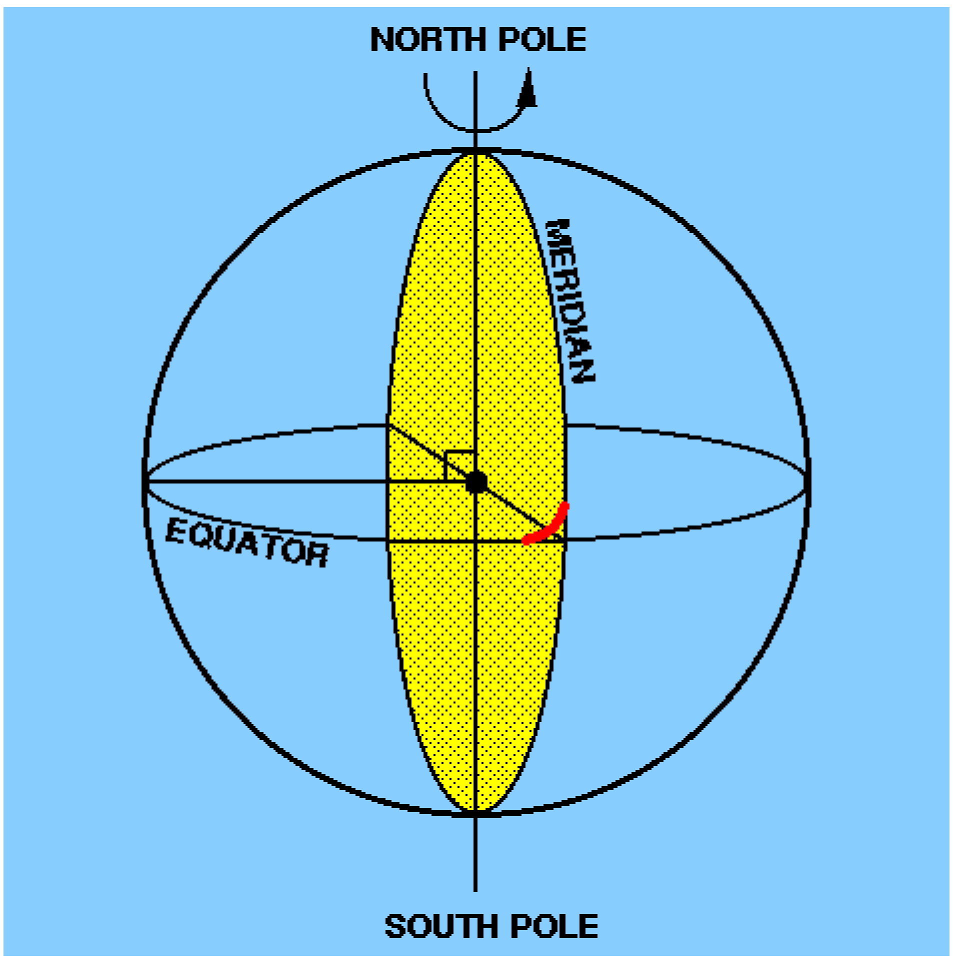
(Source: http://www.vikdhillon.staff.shef.ac.uk/teaching/phy105/celsphere/meridian.gif)

For the sake of definiteness we shall assume that the given curve is parametrized over the closed interval $[-2\pi, 2\pi]$, with the portion over $[-2\pi, 0]$ corresponding to the equator and the portion over $[0, 2\pi]$ corresponding to the fundamental meridian pair. We shall extend this to a curve parametrized over $[-4\pi, 4\pi]$ by letting the piece over $[-4\pi, 0]$ correspond to going around the equator twice and letting the piece of $[0, 4\pi]$ correspond to going around the fundamental meridian pair twice.

The next step is to modify this curve near $t = 0$ to obtain a regular smooth curve φ on the sphere which agrees with the original one off some small interval $(-h, h)$ where h is much less than $\pi/2$. Roughly speaking, the idea is to smooth out the **90** degree corner of the original curve at the point where $t = 0$. This construction is very similar to the one described in the proposition on page 3 of the following online document**:**

http://math.ucr.edu/~res/math205A/nicecurves.pdf

In terms of the previous illustration, the corner at $t = 0$ is smoothed out using the red piece in the illustration below:



We shall now use the regular curve constructed in the preceding step to define a smooth submersion from a rectangular box in the plane to the sphere. More generally, this can be done for an arbitrary regular smooth curve on the sphere, so let $\gamma(t)$ be an arbitrary curve of this sort. For the sake of simplicity assume that this curve is parametrized so that the tangent vector always has length **1.** In our example, we can in fact do this so that this new parametrization is essentially equal to the given one near the subintervals $[-4\pi, -h]$ and $[h, 4\pi]$; the only difference is that the parametrization intervals are shifted by some relatively small amount. So assume now that $\gamma(t)$ is defined on some interval (a, b). Since this curve lies on the sphere, we know that the for all values of t the vectors $\gamma(t)$ and $\gamma'(t)$ are perpendicular vectors of unit length, so that their cross product $\beta(t)$ is a unit vector which is perpendicular to both of them. Define a map σ from the open rectangular region $(a, b) \times (-\pi/2, \pi/2)$ by the formula

$$\sigma(t, u) \;=\; (\cos u)\gamma(t) \;+\; (\sin u)\beta(t).$$

The partial derivatives of this function with respect to the first and second variables are given by the following formulas:

$$\sigma_1(t, u) \;=\; (\cos u)\gamma'(t) \;+\; (\sin u)\beta'(t), \quad \sigma_2(t, u) \;=\; (-\sin u)\gamma(t) \;+\; (\cos u)\beta(t)$$

CLAIM: ***If*** $\varepsilon > 0$ ***is sufficiently small, then the two partial derivatives displayed above are linearly independent on the set*** $[a - \varepsilon, b + \varepsilon] \times (-\delta, \delta)$ ***for some*** $\delta > 0$***. Also, if*** $\gamma(t)$ ***is a (reparametrized) great circle curve for*** $c - \eta < t < c + \eta$***, then the two partial derivatives are linearly independent on the set*** $(c - \eta, c + \eta) \times (-\pi/2, \pi/2)$***.***

To see this, first note that the second partial derivative is perpendicular to $\gamma'(t)$ because it is a linear combination of nonzero orthonormal vectors with this property, and it is also nonzero. If $s = 0$, then the first partial derivative is a nonzero multiple of $\gamma'(t)$ and it follows that the two first partial derivatives at $(t, 0)$ are linearly independent, so that their cross product is nonzero; but if this is true, then by continuity of the cross product we know that the latter is also nonzero at (v, u) if v is sufficiently close to t and u is sufficiently close to 0. Basic results from point set theory then imply that the cross product of the partial derivatives is also nonzero on a set of the form $[a - \varepsilon, b + \varepsilon] \times (-\delta, \delta)$ for some $\delta > 0$.

Now suppose that we know that the restriction of the curve to $(c - \eta, c + \eta)$ is a great circle curve. In this case there is a fixed 2 – dimensional vector subspace W through the origin (namely, the span of $\gamma(t)$ and $\gamma'(t)$) such that $\gamma(t)$ and $\gamma'(t)$ lie in W. It follows that the vectors $\beta(t)$ are all unit vectors which are perpendicular to W, and by continuity it follows that $\beta(t)$ must be constant, so that $\beta'(t) = 0$. Therefore in this case we also know that the two first partial derivatives are linearly independent on the subset $(c - \eta, c + \eta) \times (-\pi/2, \pi/2)$. This completes the proof of the claim.

We shall now apply the preceding discussion to the map σ obtained from the curve φ that was constructed above. It follows that there is some $\delta > 0$ such that the restriction of σ to the region

$$U \;=\; (-4\pi, -h) \times (-\pi/2, \pi/2) \;\cup\; (-2h, 2h) \times (-\delta, \delta) \;\cup\; (h, 4\pi) \times (-\pi/2, \pi/2)$$

is a submersion. By construction, the restriction of σ to $(-4\pi, -2\pi] \times (-\pi/2, \pi/2)$ maps the latter onto all points of the 2 – dimensional sphere except for the north and south poles $(0, 0, 1)$ and $(0, 0, -1)$. Likewise, the restriction of σ to $[2\pi, 4\pi) \times (-\pi/2, \pi/2)$ maps the latter onto all points of the 2 – dimensional sphere except for $(0, 1, 0)$ and $(0, -1, 0)$. Combining these observations, we see that the restriction of σ to U is a smooth submersion whose image is the entire $2-$ sphere.

CLAIM: ***There is a subregion V of U such that V contains both of the rectangular regions $(-4\pi, -2h) \times (-\pi/2, \pi/2)$ and $(h, 4\pi) \times (-\pi/2, \pi/2)$, and V is diffeomorphic to $\mathbb{R}^2$.***

The proof of this result is based upon basic properties of bump functions which are established in Mathematics 205C; these are described in Section II.3 on pages 57 – 60 of the following online document:

http://math.ucr.edu/~res/math205C/lectnotes.pdf

Specifically, one starts with an infinitely differentiable positive real valued function g such that $g = \pi/2$ on $(-4\pi, -2h]$ and $[2h, 4\pi)$, the function g is decreasing on $[-2h, -h]$ and increasing on $[h, 2h]$, and $g = \delta/2$ on $(-h, h)$. The region V consists of all (t, u) such that $|u| \leq g(t)$. In this context we choose h such that $2h$ is much less than, say, $\pi/8$; since we can take h to be arbitrarily small, such a choice is possible.

To complete the proof of the claim, we need to prove that V is diffeomorphic to $\mathbb{R}^2$. The first step is to prove that V is diffeomorphic to $(-4\pi, 4\pi) \times (-\pi/2, \pi/2)$. An explicit "vertical" diffeomorphism from the latter to V is given by the map sending (t, u) to $\left(t,\ g(t)\cdot u\right)$; the inverse map sends (t, v) to $\left(t,\ g(t)^{-1}\cdot v\right)$. The final step is to prove that every rectangular open set of the form $(-a, a) \times (-b, b)$ is diffeomorphic to $\mathbb{R}^2$. In fact, it will suffice to show that each factor $(-a, a)$ is diffeomorphic to $\mathbb{R}$, and a standard diffeomorphism of this type is given by the map from $\mathbb{R}$ to $(-a, a)$ sending x to $(2a/\pi) \cdot \mathbf{Arctan}\ x$.