Connectedness and partial differentiation

Here is a simple application of connectedness to a basic issue in multivariable calculus.

PROPOSITION. *Let U be a nonempty connected open subset of $\mathbb{R}^n$, let $f : U \to \mathbb{R}$ be a function with continuous first partial derivatives on $\mathbb{R}$, and suppose that $\nabla f = \mathbf{0}$ on U. Then f is constant.*

Note that this result does not hold if U is disconnected; in this case all we can say is that f is constant on each connected component, and in fact we can define a function satisfying the gradient condition which is 1 on one component and 0 on the other(s).

Proof. We first show that f is locally constant. Let $\mathbf{x} \in U$, and choose $\varepsilon > 0$ such that $N_\varepsilon(\mathbf{x}) \subset U$. Given $\mathbf{y} \in U$ let $\gamma(t) = \mathbf{x} + t(\mathbf{y} - \mathbf{x})$, and let $h(t) = f \circ \gamma(t)$. By the Chain Rule for partial differentiation we have

$$h'(t) \quad = \quad \nabla f\big(\gamma(t)\big) \cdot \gamma'(t)$$

and since $\nabla f = \mathbf{0}$ it follows that $h' = 0$, which means that h is constant. Therefore we have $f(\mathbf{y}) = h(1) = h(0) = f(\mathbf{x})$. Since $\mathbf{y}$ was arbitrary this means that f is constant on $N_\varepsilon(\mathbf{x})$.

Now define an equivalence relation such that $\mathbf{u} \sim \mathbf{v}$ if and only if $f(\mathbf{u}) = f(\mathbf{v})$. By the preceding paragraph it follows that the equivalence classes of this equivalence relation are open. As usual, it follows that the union of all equivalence classes except one is also open, so that the equivalence classes are also closed. Since U is not empty there is at least one equivalence class, and since this equivalence class is open and closed the connectedness of U implies that this equivalence class is all of U. In other words, the value of f is the same at all points of U.■