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Mthematics 205C, Spring 2005, Examination 1

Each problem is worth 25 points.

1. Suppose that X is a Hausdorff topological space that is a union of two open subsets U and V, where both U and V are topological n manifolds. Prove that X is also a topological n-manifold.

SOLUTION:

Let $x \in X$, so that either $x \in U$ or $x \in V$. If $x \in U$ then there is an open subset $W \subset U$ such that $x \in W$ and W is homeomorphic to an open subset of $\mathbf{R}^n$. Since W is open in U and U is open in X, it follows that W is also open in X. Likewise, if $x \in V$, then there is an open subset $W \subset V$ such that $x \in W$ and W is homeomorphic to an open subset of $\mathbf{R}^n$. Thus the existence of W with the given property is true in all cases. Since X is assumed to be Hausdorff, it follows that X must be a topological n-manifold.■

2. Let U be an open subset of $\mathbf{R}^n$, let $x \in U$, and let V be an open subset of U with compact closure $\overline{V}$ such that $x \in V \subset \overline{V} \subset U$. Prove that there is a smooth map $f : U \to \mathbf{R}$ such that $f(x) = 1$ and $f|U - V$ is identically zero.

SOLUTION:

Choose $r > 0$ such that the disk $N_{2r}(x) \subset V$. Let φ be a smooth function that is equal to 1 for $t \leq 1$ and 0 for $t \geq \frac{3}{2}$. Consider the smooth function h_0 defined by

$$h_0(y) \quad = \quad \varphi\left(\frac{|y-x|}{r}\right) .$$

Then $f = h_0|U$ has the desired properties, for $f(x) = h(0) = 1$ and $y \notin V \Longrightarrow y \notin N_{2r}(x)$ so that

$$s \quad = \quad \frac{|y-x|}{r} \quad \geq \quad 2$$

and hence $f(y) = h(s) = 0.$∎

3. Let $(M, \mathcal{A})$ be a smooth n-manifold where $\mathcal{A}$ is a maximal smooth atlas for M.

(i) Describe a smooth atlas on $M \times M$ (you do not need to prove it is a smooth atlas).

SOLUTION:

Take a smooth atlas $\mathcal{A} = \{ (U_\alpha, h_\alpha) \}$ for M and consider all charts of the form

$$(U_\alpha \times U_\beta,\, h_\alpha \times h_\beta)$$

where α and β run independently through the indexing set for $\mathcal{A}$.■

(ii) If $\Delta_M : M \to M \times M$ is the diagonal map defined by $\Delta_M(x) = (x, x)$, explain why Δ_M is smooth.

SOLUTION:

A map into a product is smooth if and only if its projections onto two factors are smooth. Let π_1 and π_2 be the projections from $M \times M$ to the first and second factors respectively. Since $\Delta_M(x) = (x, x)$ it follows that $\pi_1 \circ \Delta_M = \pi_2 \circ \Delta_M = \mathrm{id}_M$. Since the identity map is smooth, each of these projections is smooth, and therefore Δ_M is also smooth.■

4. Suppose that $1 \leq r \leq \infty$ and we are given a smooth $\mathcal{C}^r$ homeomorphism from $\mathbf{R}^3$ to itself of the form $f(x, y, z) = (x + g(y, z), y + h(z), z)$ for some (necessarily smooth $\mathcal{C}^r$) functions g and h. Prove that f is a $\mathcal{C}^r$ diffeomorphism.

SOLUTION:

We need to show that the inverse of f is smooth. The latter is true if for all y the restriction of f^{-1} to some open neighborhood of y is smooth, and by the Inverse Function Theorem this will be true if and only if the derivative is invertible at every point of $\mathbf{R}^3$. The latter holds if and only if the Jacobian at each point is nonzero, so it suffices to show that the Jacobian is nonzero. But direct computation shows that this Jacobian is equal to

$$\begin{vmatrix} 1 & g_y & g_z \\ 0 & 1 & h_z \\ 0 & 0 & 1 \end{vmatrix} = 1$$

and thus the hypothesis of the Inverse Function Theorem is satisfied. By the previous discussion this implies that f^{-1} is smooth.■