EXERCISES FOR MATHEMATICS 205C

SPRING 2005

Exercises on cotangent bundles, riemannian metrics and exterior forms

V. Cotangent spaces and tensor algebra

V.A : Dual spaces

(Conlon, § 6.1)

Additional exercises

1. Let V be a finite dimensional vector space over the field $\mathbf{k}$, and let W_1 and W_2 be subspaces of V. Define the annihilator $W^\dagger \subset V^*$ of a subspace $W \subset V$ as in the notes. Prove that the following hold:

(i) $$W_1 \subset W_2 \implies W_2^\dagger \subset W_1^\dagger$$

(ii) $$(W_1 + W_2)^\dagger = W_1^\dagger \cap W_2^\dagger$$

(iii) $$(W_1 \cap W_2)^\dagger = W_1^\dagger + W_2^\dagger$$

To what extent do these remain valid if V is infinite-dimensional?

2. Let V and W be finite-dimensional vector spaces over a field $\mathbf{k}$, and let $T : V \to W$ be a linear transformaion.

(i) Prove that the kernels and images of T and T^* satisfy the following identities:

$$\text{Kernel}(T^*) = \big(\text{Image}(T)\big)^\dagger \qquad \text{Image}(T^*) = \big(\text{Kernel}(T)\big)^\dagger$$

(ii) Using

$$(i)$$

, prove that the ranks of T and T^* are equal (without using matrices!), and find a formula for

$$\dim(\text{KernelT}) - \dim(\text{KernelT}^*)$$

that only involves $\dim V$ and $\dim W$.

3. Let V be an n-dimensional real inner product space with inner product $\langle\ ,\ \rangle$, and let $\mathcal{A} = \{\,\mathbf{a}_1, \cdots, \mathbf{a}_n\,\}$ be an ordered basis for V.

(i) Prove that there is a unique *adjoint basis* for V of the form $\mathcal{A}^* = \{\,\mathbf{a}_1^*, \cdots, \mathbf{a}_n^*\,\}$ such that $\langle \mathbf{a}_i^*, \mathbf{a}_j \rangle$ is equal to 0 if $i \neq j$ and 1 if $i = j$.

(*ii*) If $\mathcal{A}$ and $\mathcal{A}^*$ are as above and $\mathbf{x} \in V$, prove that

$$\mathbf{x} = \sum_{j=1}^{n} \langle \mathbf{x},\, \mathbf{a}_j^* \rangle\, \mathbf{a}_j \quad = \quad \sum_{j=1}^{n} \langle \mathbf{x},\, \mathbf{a}_j \rangle\, \mathbf{a}_j^* \ .$$

(*iii*) Suppose that $V = \mathbf{R}^n$ with the standard inner product and the vectors of $\mathcal{A}$ are given by the columns of the invertible matrix P. If X denotes the corresponding invertible matrix whose columns display the vectors in the adjoint basis $\mathcal{A}^*$, what is the relationship between P and Q? Prove that your formula is correct.

(*iv*) Suppose that we are given another inner product for $\mathbf{R}^n$ with Gram matrix G (the latter displays the inner products of the various pairs of standard unit vectors $\mathbf{e}_i$ and $\mathbf{e}_j$). What is the relation between P and Q in this case? Once again, prove your formula is correct. [*Hint:* The new inner product can be evaluated using the matrix product ${}^{\mathbf{T}}\mathbf{y}\, G\, \mathbf{x}$.]

V.1 : Vector bundles

(Conlon, §§ 3.3–3.4)

Conlon, p. 102: 3.4.18

Additional exercises

1. Let $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$. Prove that the standard inclusions of $\mathbf{kP}^n$ in $\mathbf{kP}^{n+1}$ sending a point with homogeneous coordinates $(x_1, \cdot, x_{n+1})$ to coordinates $(x_1, \cdot, x_{n+1}, 0)$ is a smooth embedding.

15. Suppose that two vector bundles ($\pi : E \to B$, *etc.*) and ($\pi' : E' \to B$, *etc.*) are isomorphic, and let $z : B \to E$ $z' : B \to E'$ be the respective zero sections. Prove that $E - z(B)$ is homeomorphic to $E' - z'(B)$.

25. Show that the functions $g_{1,1} = y^4 + y^2 + 2xy + x^3 + 1$, $g_{1,2} = g_{2,1} = y + xy^2 + 2x$, $g_{2,2} = 2x^2 + 1$ define a riemannian metric on $\mathbf{R}^2$.

26. Show that the functions $g_{1,1} = 2$, $g_{1,2} = g_{2,1} = x$, $g_{2,2} = x^2 + 1$, $g_{2,3} = g_{3,2} = x$, $g_{1,3} = g_{3,1} = y$, $g_{3,3} = x^2 + y^2 + 1$ define a riemannian metric on $\mathbf{R}^3$.

10. Prove that a 1-dimensional real vector bundle ξ over a simply connected manifold M is a trivial vector bundle. [*Hint:* Put a riemannian metric on the bundle, and let $S(\xi)$ be its unit sphere bundle. Why is the complement of the zero section homeomorphic/diffeomorphic to $S(\xi) \times \mathbf{R}$? Show that $S(\xi)$ is a 2-sheeted covering space over M and thus splits into two pieces, each homeomorphic to M, and that either of these determines a nowhere zero continuous/smooth cross section. Use this cross section to construct a bundle isomorphism from ξ to the 1-dimensional product bundle.]

V.2 : Cotangent spaces and differential 1-forms

(Conlon, §§6.1–6.2)

14. Let ($\pi : E \to B$, *etc.*) be a smooth vector bundle projection, and let $z : B \to E$ be the zero section. Prove that the identity map of E is smoothly homotopic to $z \circ \pi$. [*Hint:* How can you prove this if B consists of a single point?]

17. Let U be an open subset in Euclidean n-space, and let g be a riemannian metric on M. Prove that there is a set of n vector fields over U that are orthonormal with respect to g. Prove also that there is a vector bundle automorphism of U (*i.e.*, a homeomorphism Ψ from $U \times \mathbf{R}^n$ to itself such that for each $u \in U$ Ψ maps $\{u\} \times \mathbf{R}^n$ to itself by an invertible linear transformation) such that Ψ sends g to the trivial metric. In other words

$$g\left(\Psi((u.\ \mathbf{v}))\,,(\Psi((u.\ \mathbf{w}))\right) = \langle \mathbf{v},\ \mathbf{w}\rangle.$$

18. The Poincaré metric on the upper half plane $\mathbf{H}_+ = \{x + iy \mid y > 0\}$ is defined by the formula

$$g_{\mathbf{H}} = \frac{dx^2 + dy^2}{y^2}\ .$$

Given a 2×2 real matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with determinant $ad - bc = 1$, let F_A be defined by the formula

$$F_A(z) = \frac{az + b}{cz + d}$$

where the point z lies in the upper half plane and the right hand side is interpreted using complex numbers. Prove that F_A defines a diffeomorphism of the upper half plane to itself and that the associated map of tangent spaces is an isometry with respect to the Poincaré metric.

19. The Poincaré metric on the open disk $\mathbf{D} = \{x + iy \mid x^2 + y^2 < 1\}$ is given by the formula

$$g_{\mathbf{D}} = \frac{dx^2 + dy^2}{\left(1 - (x^2 + y^2)^2\right)}.$$

Let $f : \mathbf{D} \to \mathbf{H}_+$ be the complex analytic map

$$f(z) = i\frac{1 - z}{1 + z}.$$

Prove that f is a diffeomorphism and sends the Poincaré metric on $\mathbf{D}$ to the Poincaré metric on $\mathbf{H}_+$; *i.e.*, if $\mathbf{v}$ and $\mathbf{w}$ are tangent vectors over the same point x then

$$g_{\mathbf{D}}(\mathbf{v},\ \mathbf{w}) = g_{\mathbf{H}}\left(T(f)\mathbf{v},\ T(f)\mathbf{w}\right).$$

V.3 : Line integrals

(Conlon, (Conlon, §6.3)

Additional exercises

1. discrete.

V.4 : Tensor and exterior products

(Conlon, §§7.1–7.2, 7.4)

Conlon, pp. 225–226: 7.2.21, 7.2.23

Additional exercises

1. Let U, V, W be finite-dimensional real vector spaces. Prove the following relationships:

(a) $(U \oplus V) \otimes W \cong (U \otimes W) \oplus (V \otimes W)$.

(ii) $U^* \otimes V^* \cong (U \otimes V)^*$.

(iii) $\mathrm{Hom}(U, V) \cong U^* \otimes V$.

2. If U is an open subset of $\mathbf{R}^n$ and $\mathcal{X}(U)$ is the space of smooth vector fields on U, then the general considerations about tensor fields show that the identity map on $\mathcal{X}(U)$ defines a tensor field $\mathbf{K}$ of type $(1,1)$ on U. The latter can be written in the form

$$\mathbf{K}(u) = \sum_{i,j} b_i^j(u)\, dx^i \otimes \frac{\partial}{\partial x^j}$$

for $b_i^j \in C^\infty(M)$. What are the functions b_i^j? [*Hints:* Look at **1.**

(iii)

above; each function can be written down with a very small number of symbols.]

V.5 : Constructing tensor fields

(Conlon, §7.5)

Additional exercises

1. discrete.

VI. Spaces with additional properties

VI.1 : Exterior differential calculus

(Conlon, § 8.1)

Munkres, § 30, pp. 194–195: 9 (first part only), 10, 13*, 14*

Additional exercises

1. If $(X, \mathbf{T})$ is a second countable Hausdorff space, prove that **2.** Consider the 2-form on $\mathbf{R}^3 - \{\mathbf{0}\}$

$$\omega = \frac{1}{r} x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy$$

where $r^2 = x^2 + y^2 + z^2$. Show that ω is not exact and that $dr \wedge \omega = dx \wedge dy \wedge dz$.

4. Let $\alpha = x\, dx + y\, dy + z\, dz$ and $\Omega = dx \wedge dy \wedge dz$ be differential forms on $\mathbf{R}^3$. Write down a differential form β on $\mathbf{R}^3 - \{\mathbf{0}\}$ such that $\Omega = \alpha \wedge \beta$, and show that there is no differential form γ on $\mathbf{R}^3$ such that $\Omega = \alpha \wedge \gamma$.

10. Prove that if ω is a 1-form then $\omega \wedge \omega = 0$. Give an examples to show the analog is false for higher degree forms by exhibiting a 2-form ω on $\mathbf{R}^{2n}$ such that the n-fold wedge $\wedge^n \omega = \omega \wedge \cdots \omega \neq 0$ at every point.

6. Let ω be a nowhere zero smooth 1-form on a smooth compact manifold M. Show that if $\omega \wedge d\omega = 0$ then there exists a 1-form α such that $d\omega = \alpha \wedge \omega$ [*Hint:* First do it locally then use a partition of unity.]

7. Consider a closed 2-form on $\mathbf{R}^3 - \{\mathbf{0}\}$ defined by

$$\omega = P(x, y, z)\, dy \wedge dz + Q(x, y, z)\, dx \wedge dz + R(x, y, z)\, dx \wedge dy$$

where P, Q and R are all smooth functions. Let $r^2 = x^2 + y^2 + z^2$, and assume that

$$|P|,\ |Q|,\ |R|\ < \frac{1}{r}.$$

Show that ω is exact.

8. Let M be a second countable smooth manifold and let $f : M \to \mathbf{R}$ be a smooth function such that the exterior derivative df is nowhere zero. Prove that M is noncompact, and using partitions of unity show that there is a smooth vector field X such that $\langle df, X \rangle = 1$ at all points of M.

9. Let ω be the 1-form on $\mathbf{R}^3$ defined by $x\, dy - y\, dx + dz$. Show that for every nonzero real valued function f the form $f\omega$ is not closed.

VI.2 : Orientability

(Conlon, § 3.4)

Munkres, § 28, pp. 181–182: 6

Additional exercises

1.* Let X be a compact Hausdorff space, let Y be a Hausdorff space,

VI.3 : The Poincaré Lemma

(Conlon, § 8.3)

Munkres, § 26, pp. 170–172: 11*

Additional exercises

1. If $(X, \mathbf{T})$ is compact Hausdorff and $\mathbf{T}^*$ is strictly

VI.4 : Generalized Stokes' Formula

(Conlon, § 8.3)

Munkres, § 38, pp. 241–242: 2*, 3 (just give a necessary condition on the topology of the space)

Additional exercises

1. uoiupoiypioy

3. Let $f : S^2 \to \mathbf{R}^2$ be smooth, and let ω be a 2-form on $\mathbf{R}^2$. Prove that the integral of the pulled back form over the sphere is zero, and prove that there must be a point on the sphere where the pullback vanishes.

4.

VI.5 : de Rham cohomology

(Conlon, §§ 6.4, 8.4–8.6)

Munkres, § 40, p. 252: 2, 3

Additional exercises

1. A *pseudometric space* is a pair $(X, \mathbf{d})$ consisting of

5. Let $M = \mathbf{R}^3 - (X \cup Y)$, where X and Y denote the x- and y-axes respectively. Find closed 1-forms representing a basis for the first de Rham cohomology group $H^1_{\mathrm{DR}}(M)$.

VI.6 : de Rham's Theorem

(Conlon, § 8.9, Appendix D.1–D.4)

Munkres, § 40, p. 252: 2, 3