More details about integral flows to vector fields

This document contains additional details about the structure of integral flows, and in particular it contains proofs that

(1) maximal integral curves exist,

(2) the integral flow is in fact defined on an open subset of $\mathbf{R} \times M$.

In the discussion below M will denote a smooth manifold and X will denote a smooth vector field on M.

Proof of (1). Suppose that we are given two smooth curves $\alpha_0,\ \alpha_1$ from open intervales $J_0,\ J_1$ to M such that each is an integral curve for the differential equation, and suppose that $\alpha_0(t_0) = \alpha_1(t_0)$ for some $t_0 \in J_0 \cap J_1$. We claim that $\alpha_0|J_0 \cap J_1 = \alpha_1|J_0 \cap J_1$. The intersection of the intervals is itself an open interval, and the set of points where $\alpha_0 = \alpha_1$ is closed by general topological considerations. Furthermore, by the local uniqueness results for solutions of ordinary differential equations the set of points where $\alpha_0 = \alpha_1$ is open. Therefore by connectedness the intersections of α_0 and α_1 to $J_0 \cap J_1$.

It follows that if $\{\,\gamma_\alpha\,\}$ is a collection of integral curves for X such that $\gamma_\alpha(0) = x_0$ then their union defines an integral curve, and this must be a maximal integral curve with initial condition x_0.■

Proof of (2). Consider the family of all open sets $W \subset \mathbf{R} \times M$ for which a smooth flow can be defined on M such that each intersection $W \cap \mathbf{R} \times \{p\}$ has the form $(a, b) \times \{p\}$ for some open interval (a, b) containing 0. The union of these open subsets is the maximal domain for a smooth flow function. We shall call this flow function Φ and denote its domain of definition by $\mathcal{D}(X)$.

We need to show that for all $p \in M$ the curve $\Phi|\mathcal{D}(X) \cap \mathbf{R} \times \{p\}$ is a maximal curve. As in the preceding paragraph let $W \cap \mathbf{R} \times \{p\} = (a, b) \times \{p\}$. If the integral curve is not maximal then either $b < +\infty$ and the maximal integral can be defined for parameter value b or else $a > -\infty$ and the maximal integral can be defined for parameter value a. We shall show that the integral curve cannot be extended to parameter value a in the first case; the proof in the other case is similar and will be left to the reader.

Since the integral curve γ with initial condition p can be defined for parameter value b, if $q = \gamma(b)$ then one can define a smooth partial flow

$$\beta : (a - \eta,\ a + \eta) \times V \longrightarrow\ M$$

such that $p \in V$ and $\beta|(b - \eta, b + \eta) \times \{p\}$ is an integral curve for X with initial condition q. If necessary we may replace η with a smaller positive value to ensure that γ maps $(b - \eta, b + \eta)$ to V. Choose c and $\delta > 0$ such that $b - \eta < c < b$, $(c, p) \in \mathcal{D}(X)$ and Φ maps $(c - \delta,\ c + \delta) \times W$ into V for some open neighborhood W of p in M. We then form the function

$$\alpha(t, x) \quad = \quad \beta(t - c,\ \alpha(c, x)\,)|(c - \eta,\ c + \eta) \times W\ .$$

It follows that the restrictions of Φ and α to appropriate subintervals of $\mathbf{R} \times \{w\}$ are integral curves for X with the same values at c, and therefore the restrictions agree on the open subsets on which both are defined. This implies that the union of α and Φ is a smooth flow defined on an open subset containing both $\mathcal{D}(X)$ and the point (b, p); since the latter was not supposed to be a point of $\mathcal{D}(X)$ and $\mathcal{D}(X)$ was assumed to be maximal, we have a contradiction. The problem arises from our assumption that (b, p) does not lie in $\mathcal{D}(X)$ but the integral curve with initial condition p is defined for parameter value b, and therefore it is not possible to define the integral curve for parameter value b.

As noted before, a similar argument applies if $a > -\infty$ to show that the integral curve cannot be extended to parameter value a.■