

## Nonsubmersion examples for compact manifolds

On page 129 of `lectnotes.pdf` we noted that regular homotopy classes of smooth submersions from a **noncompact** connected manifold  $M^m$  to another manifold  $N^n$  (where  $m \geq n$ ) are in 1-1 correspondence with homotopy classes of continuous tangent bundle morphisms  $F : T(M) \rightarrow T(N)$  satisfying the following conditions:

- (i) There is a continuous map  $f : M \rightarrow N$  such that  $\tau_N \circ F = f \circ \tau_M$ .
- (ii) For each  $x \in M$ , the induced map  $F_x$  from  $T_x(M)$  to  $T_{f(x)}(N)$  is a linear surjection.

Furthermore, we mentioned that this classification does not extend to submersions defined on **compact** manifolds, and we described a class of pairs  $(F, f)$  such that properties (i) and (ii) hold but there is no smooth submersion  $g : M \rightarrow N$  such that  $f$  is homotopic to  $g$ .

Our purpose here is to describe some similar, but simpler, examples for which we can give a self-contained proof that the underlying map  $f : M \rightarrow N$  is not homotopic to a smooth submersion; in fact, in these examples  $f$  is not even homotopic to a **topological** submersion, and it is also possible to prove a homotopy-theoretic analog of these results involving a concept of **compact fiberings** (however, we shall not prove these generalizations).

Despite the file name for this note, our arguments do not require C. Ehresmann's theorem that a proper smooth submersion is a smooth fiber bundle projection (see the file `ehresmann.pdf` in the course directory). In fact, our arguments only use material at the level of this course and its two prerequisites.

### *The examples*

For each positive integer  $k$ , let  $T^k$  denote the Cartesian product of  $k$  copies of the circle. Since the tangent bundle  $\mathbf{T}(S^1)$  is isomorphic to the trivial vector bundle  $S^1 \times \mathbf{R}^1$  and the tangent bundle of a product satisfies  $\mathbf{T}(A \times B) \cong \mathbf{T}(A) \times \mathbf{T}(B)$ , it follows that for each  $k$  we have  $\mathbf{T}(T^k) \cong T^k \times \mathbf{R}^k$ .

Suppose now that  $m \geq n > 0$ , and let  $f : T^m \rightarrow T^n$  be the constant map. If  $C : \mathbf{R}^m \rightarrow \mathbf{R}^n$  is an arbitrary linear transformation of maximum rank (for example, projection onto the first  $n$  coordinates), then one can form a map  $F$  satisfying properties (i) and (ii) with respect to  $f$  by taking the mapping

$$f \times C : T^m \times \mathbf{R}^m \longrightarrow T^n \times \mathbf{R}^n$$

and composing it with the isomorphisms

$$\mathbf{T}(T^m) \cong T^m \times \mathbf{R}^m \qquad \mathbf{T}(T^n) \cong T^n \times \mathbf{R}^n$$

described above.

Having shown the existence of a tangent bundle surjection  $F$  such that  $\tau_N \circ F = f \circ \tau_M$ , we shall have our examples if we can prove the following result.

**Proposition.** *If  $m \geq n > 0$ , then the constant map  $f$  from  $T^m$  to  $T^n$  is not homotopic to a smooth submersion.*

In the course of the proof we shall need the following elementary fact relating basepoint preserving homotopies and homotopies that are not necessarily basepoint preserving.

**Lemma.** *Let  $(X, x)$  be a topological space, let  $G$  be a connected topological group, and take the identity for the basepoint of  $G$ . Then two basepoint preserving maps from  $(X, x)$  to  $(G, 1)$  are basepoint preservingly homotopic if and only if they are homotopic by a homotopy which is not necessarily basepoint preserving.*

This applies to our examples because every torus is a connected topological group.

**Proof of the lemma.** The  $(\Rightarrow)$  implication is trivial, so it is only necessary to discuss the opposite implication. Let  $H$  be a homotopy which is not necessarily basepoint preserving, and define a closed continuous curve  $\gamma(t) = H(x, t)$ . Then  $K(y, t) = H(y, t) \cdot \gamma(t)^{-1}$  defines a basepoint preserving homotopy relating the two original mappings. ■

**Proof of the proposition.** Suppose that  $f$  is homotopic to some smooth submersion, say  $g_0$ . Our first step will be to show that  $g_0$  is homotopic to a basepoint preserving smooth submersion. This is fairly easy to do. Since  $T^k$  is arcwise connected, there is a continuous curve  $\alpha$  joining 1 to  $g_0(1)$ , and since the group operation on  $T^k$  is smooth it follows that translation by a group element is a diffeomorphism; therefore the map  $g_0(z) \cdot \alpha(t)^{-1}$  defines a regular homotopy from  $g_0$  to a smooth submersion which is basepoint preserving. Let  $g$  denote this basepoint preserving submersion.

By the lemma we know that  $g$  is basepoint preservingly homotopic to the constant map, and therefore the associated map of fundamental groups

$$g_* : \pi_1(T^m) \rightarrow \pi_1(T^n)$$

is trivial. Therefore, the lifting criterion implies the existence of a continuous mapping  $h : T^m \rightarrow \mathbf{R}^n$  such that  $g = p \circ h$ , where  $p : \mathbf{R}^n \rightarrow T^n$  is the usual simply connected covering space projection. In fact, since the covering space projection  $p$  is a smooth submersion, it follows that  $h$  must also be a smooth submersion. Since smooth submersions are open mappings, it follows that the image of  $h$  must be **open** in  $\mathbf{R}^n$ .

On the other hand, since  $T^m$  is compact, it follows that the image of  $h$  must be a **compact** subset of  $\mathbf{R}^n$ . This is impossible because  $\mathbf{R}^n$  has no nonempty subsets that are both compact and open. The source of the contradiction is our assumption that the constant map is homotopic to a smooth submersion, and therefore no such mapping can exist, which is exactly what we wanted to prove. ■