One-parameter groups of linear ordinnary differential equations

Let A be and $n \times n$ matrix, and consider the vector field $\mathbf{F}_A$ on $\mathbf{R}^n$ defined by

$$\mathbf{F}_A(u) = (u, Au) \in \mathbf{R}^n \times \mathbf{R}^n .$$

In some undergraduate differential equations courses it is shown that the integral curves for this vector field have the form

$$\exp(tA)u_0$$

where the matrix exponential is defined by the usual sort of power series

$$\exp(B) = \sum \frac{1}{k!} B^k .$$

It turns out that this power series converges for all choices of B.

Here are some useful properties of the exponential function on matrices:

(i) If B and C are commuting square matrices, then $\exp(B+C) = \exp(B)\exp(C)$. [*Warning:* This is not necessarily true if B and C do not commute!]

(ii) If D is a diagonal matrix with diagonal entries $d_1, \cdots d_n$, then $\exp(D)$ is a diagonal matrix whose entries are $e^{d_1}, \cdots e^{d_n}$.

(iii) If A is an $n \times n$ matrix and P is an invertible $\times n$ matrix, then one has $\exp(P^{-1}AP) = P^{-1}\exp(A)P$.

(iv) If A is a nilpotent matrix satisfying $A^2 = 0$, then $\exp(A) = I + A$. More generally if $A^n = 0$, then $\exp(A)$ is a polynomial in A of degree $\leq n-1$.

One can use the preceding together with the Jordan canonical form to write down the exponential function $\exp(tA)$ explicitly as a matrix of functions of t. One particularly useful example to consider is the 2×2 matrix $A = 2I + N$ where N has a 1 in the upper right hand corner and zeros elsewhere.

Applications to one-parameter groups. A typical problem is to consider the global one-parameter group φ_t^A associated to the vector field $\mathbf{F}_A$ described at the beginning of this note. For example one might ask for a description of $\varphi_1^A(\Gamma)$ where Γ is some standard curve—the unit circle would be a typical example.

Consider the diagonal matrix D whose diagonal entries are given by $+1$ and -1 in that order. It follows that $\exp(tD)$ is the diagonal matrix whose entries are e^t and e^{-t} in that order. What is φ_1^D of the unit circle?

Write

$$\varphi_1^D(x, y) = (u(x, y), v(x, y)) .$$

We then know that $\varphi_1^D(V) = \exp(D)V$, and consequently $u = ex$ and $v = e^{-1}y$. To find the equation of the curve satisfied by u and v we have to solve for x and y in terms of u and v and then substitute into the standard equation of the unit circle:

$$x^2 + y^2 = 1$$

In this case it is easy to do so, for $x = e^{-1}u$ and $y = ev$. Thus the corresponding equation in u and v is

$$\left(\frac{u}{e}\right)^2 + (ev)^2 = 1$$

which defines an ellipse centered at the origin and passing through the points $(\pm e, 0)$ and $(0, \pm e^{-1})$. As t ranges over the positive real numbers the images of the circle under $\exp(tD)$ are ellipses that become increasingly compressed in one direction and increasingly stretched in the other.