The Long Line

The *Long Line* is an example of a topological 1-manifold that is not metrizable. The purpose of this file is to define this space and prove it is not metrizable. This example appears in almost every graduate level point set topology text (for example, see Munkres, pp. 158–159 However, some of the basic ideas behind the example are not always covered completely in topology courses; in particular, this applies to some points about well-ordered sets. We shall begin by summarizing what we need and giving appropriate references to Munkres.

For our purposes, Section 10 of Munkres will be the basic source of background information on well-ordered sets. We shall add one *notational conventions*: If S is a well-ordered set, then 0_S (or sometimes simply 0) will denote the unique minimal element of S, and if $\alpha \in S$ is **not** a maximal element then $\alpha + 1$ will denote the least element of S that is strictly greater than α. Following standard practice we shall denote $0 + 1$ by 1, $1 + 1$ by 2, and so on.

MINIMAL UNCOUNTABLE WELL-ORDERINGS

By the Well-Ordering Principle, every set can be well-ordered. In particular, this holds for an arbitrary uncountable set, and as shown in Munkres (see Lemma 10.2 on page 66) this implies the existence of a well-ordered set S_Ω with the following two properties:

(i) *The set S_Ω is uncountable.*

(ii) *For all $\alpha \in S_\Omega$ the set*

$$\mathbf{IS}(\alpha, S_\Omega) \;=\; \{\ \beta \in S_\Omega \mid \beta \;<\; \alpha\ \}$$

is countable. (The terminology **IS** is derived from the order-theoretic description of this set as an initial segment.)

In particular, it follows that S_Ω cannot have a maximal element (otherwise it would be countable, being the union of everything less than the maximal element and the element itself).

In fact, if one develops more of the theory of well-ordered sets than appears in Munkres, then it is possible to show that such a set is unique up to an order-preserving 1–1 correspondence, but we shall not need this fact (see the topology texts by Dugundji [D] and Kelley [K] for more information in this direction — both contain more detailed treatments of ordinal numbers and well-ordered sets than Munkres).

The space S_Ω and closely related objects occur repeatedly in point set topology. They yield many counterexamples to show that certain questions about topological spaces have negative answers (*e.g.*, products and subspaces of T_4 spaces are not necessarily T_4); we shall be using S_Ω here for the same purpose. A crucial property of S_Ω for constructing counterexamples is given by Theorem 10.3 on page 66 of Munkres: *Every countable subset of S_Ω has an upper bound in S_Ω.*

Since S_Ω is well-ordered, it follows that every countable well-ordered subset has a **least** upper bound in S_Ω.

LEXICOGRAPHIC ORDERINGS

If A and B are linearly ordered sets, then one can make their cartesian product $A \times B$ into a linearly ordered set with the so-called **lexicographic** (or *dictionary*) ordering. Specifically, if we are given two ordered pairs (a, b) and (a', b') in $A \times B$, then $(a, b) < (a', b')$ if and only if **either** $a < a'$ **or else** $a = a'$ and $b < b'$. It is a routine exercise to verify that this indeed describes a linear ordering on $A \times B$, and the proof is left to the reader as an exercise.

The *Long Line* $\mathbb{LL}$ is the linearly ordered set defined by first taking the product $S_\Omega \times [0, 1)$ (where the second factor is the usual half open interval in the real line) and removing the minimal element $(0, 0)$. We make $\mathbb{LL}$ into a topological space by taking the associated order topology as defined in Section 14 of Munkres.

By construction there is a canonical strictly increasing map from $S_\Omega - \{0\}$ to $\mathbb{LL}$ sending α to $(\alpha, 0)$; we shall denote this map by j.

PROPERTIES OF THE LONG LINE

By Theorem 17.11 on page 100 of Munkres, every linearly ordered set is Hausdorff with respect to the order topology, and therefore it follows that the Long Line is Hausdorff. We shall do this by completing several steps of the argument sketched in Exercise 12 on pages 158–159 of Munkres (specifically, Steps $(a) - (c)$ from the Theorem at the top of page 159).

OBSERVATION 1. *For each $\alpha \in S_\Omega - \{0\}$ there is a $1-1$ order-preserving map from the set*

$$J(\alpha) \;=\; \{\; x \in \mathbb{LL} \mid x \;<\; j(\alpha) \;\}$$

to an open interval in the real numbers.

Given two partially ordered sets, we say that they have the same *order type* if there is a 1–1 order-preserving correspondence between them.

Proof: We go through the steps on page 159 of Munkres:

STEP (a): *If X is a linearly ordered set with $a < b < c \in X$, then the half-open interval $[a, c)$ has the order type of $[0, 1)$ if and only if the same is true for $[a, b)$ and $[b, c)$.*

To prove the $(\Longrightarrow)$ implication, it suffices to show that if $s \in (0, 1)$ then there are 1–1 order-preserving correspondences between $[0, 1)$ and each of $[0, s)$, $[s, 1)$. Such maps are given by linear functions sending 0 and 1 to 0 and s in the case of $[0, s)$ and sending 0 and 1 to s and 1 in the case of $[s, 1)$. To prove the $(\Longleftarrow)$ implication, suppose that f and g are 1–1 order-preserving correspondences between $[0, 1)$ and the respective intervals $[a, b)$ and $[b, c)$. Then one can define a 1–1 correspondence between $[0, 2)$ and $[a, c)$ whose value is $f(t)$ if $t \in [0, 1)$ and $g(t - 1)$ if $t \in [1, 2)$. Since multiplication by 2 defines a 1–1 order-preserving correspondence from $[0, 1)$ to $[0, 2)$, it follows that $[a, c)$ has the order type of $[0, 1)$.■

STEP (b): *Let X be a linearly ordered set, let $\{\, x_n \,\}$ be a strictly increasing sequence in X, and suppose that this sequence has a least upper bound that we shall denote by b. Then $[x_0, b)$ has the order type of $[0, 1)$ if and only if this is true for each interval $[x_i, x_{i+1})$.*

As in the previous step, the $(\Longrightarrow)$ implication reduces to considering the following special case: *Given a strictly increasing sequence $\{\, x_n \,\}$ in $[0, 1]$ whose limit is 1, then each interval*

$[x_i, x_{i+1})$ *has the order type of* $[0,1)$. The desired 1–1 correspondences are given by linear functions whose values at 0 and 1 are x_i and x_{i+1} respectively. To prove the ($\Longleftarrow$) implication, begin with order-preserving 1–1 correspondences $f_i : [0,1) \to [x_i, x_{i+1})$, and define a similar map

$$f : [0,\infty) \longrightarrow \cup_i \ [x_i, x_{i+1})$$

such that $f(x) = f_i(x-i)$ if $x \in [i, i+1)$; this yields a well-defined map because each nonnegative real number lies in a unique half-open interval of the form $[i, i+1)$.

Since there is a 1–1 order preserving correspondence between $[0,\infty)$ and $[0,1)$ sending t to $t/(1+t)$, the desired conclusion will follow if we can show that $[x_0, b)$ is equal to ince each x_i is contained in $[x_0, b)$ it follows that the latter contains the union described above. To prove the reverse inclusion, suppose that $y \in [x_0, b)$. Since b is the least upper bound of the sequence $\{ x_n \}$ it follows that y is not an upper bound for the set and therefore there is some x_k such that $y < x_k$. By finite induction and Step (a), it follows that $[0, x_k)$ has the order type of $[0,1)$, and the latter in turn implies that $y \in [x_i, x_{i+1})$ for some $i < k$.■

STEP (c): *For each* $\alpha \in S_\Omega - \{0\}$ *the set* $J(\alpha)$ *has the order type of* $(0,1)$.

It will suffice to show that $J(\alpha) \cup \{j(0)\}$ has the order type of $[0,1)$. Suppose that the result is false for some α; by well-ordering there is a least α^* with this property. Furthermore, we clearly must have $\alpha^* > 1$. There are now two possibilities: Either the initial segment $\mathbf{IS}(\alpha^*, S_\Omega)$ has a maximal element or it does not. **CLAIM:** In the first case, $\alpha^* = \beta + 1$ for some β, and in the second case it is the least upper bound of some strictly increasing sequence $\{ \alpha_n \}$.

In the first case, let β be the maximal element of the initial segment. Since both $\beta + 1$ and α^* are strictly greater than β, it follows that $\alpha^* \geq \beta + 1$. However, if strict inequality held, then $\beta + 1$ would also belong to the initial segment, contradicting the maximality of β. Therefore we must have $\alpha^* = \beta + 1$.

In the second case, we begin by recalling that the intial segment is countable. In fact, it must also be infinite since it has no maximal element. Let φ be a 1–1 onto mapping from the positive integers to the initial segment (note that we say nothing about φ being order-preserving, and in fact order-preserving maps cannot exist unless α^* is the **first** element with an infinite initial segment!). Consider the sequence of elements $\alpha_n < \alpha^*$ defined recursively as follows: Take α_1 to be the first element of the initial segment that is greater than $\varphi(1)$, and assuming α_k has been constructed for $k < n$ take α_n to be the first element of the initial segment that is greater than both α_{n-1} and $\varphi(n)$. The existence of these first elements is guaranteed by well-ordering and the hypothesis that the initial segment has no maximal element. We claim that α^* is the least upper bound β^* of the sequence $\{ \alpha_n \}$. Since the sequence lies in the initial segment, clearly $\alpha^* \geq \beta^*$. If these elements were strictly unequal, then we would have $\beta^* = \varphi(m)$ for some m and hence $\beta^* < \alpha_m$, contradicting the definition of β^* as the least upper bound of the sequence. Therefore α^* must be the least upper bound as claimed.

The remainer of the proof of Step (c) splits into the two cases analyzed thus far.

CASE I. If $\alpha^* = \beta + 1$ then it follows that $J(\beta) \cup \{0\}$ has the order type of $[0,1)$ and hence by (a) the same is true for $J(\alpha^*) \cup \{0\}$. But this contradicts our hypothesis, and therefore we see that α cannot have the form $\beta + 1$.

CASE II. If α^* is the least upper bound of some strictly increasing sequence $\{ \alpha_n \}$, then the minimality of α^* and (a) implies that each of the sets $J(\alpha_n) \cup \{j(0)\}$ and $J(\alpha_{n+1}) - J(\alpha_n)$ has the order type of $[0,1)$. Therefore by (b) we know that $J(\alpha^*)$ also has the order type of $[0,1)$.

Once again this contradicts our hypothesis,] and therefore we see that α cannot be the least upper bound of some strictly increasing sequence $\{ \alpha_n \}$.

The preceding proofs by contradiction show that one cannot find an $\alpha \in \mathbb{LL}$ such that $J(\alpha) \cup \{0\}$ does not have the order type of $[0, 1)$, and therefore we have shown the conclusion of (c).■

COROLLARY. *The space $\mathbb{LL}$ is a topological 1-manifold.*

Proof. We begin with the following elementary observation: *Let B be an open interval in the linearly ordered set A, and let $\mathbf{T}_B$ $\mathbf{T}_A$ denote the respective order topologies. Then U is open with respect to $\mathbf{T}_B$ if and only if it is open with respect to $\mathbf{T}_A$.* The verification is elementary and left to the reader as an exercise.

Suppose now that $(\alpha, t) \in \mathbb{LL}$ Then by construction $(\alpha, t) \in J(\alpha + 1)$. By Step (c) above and the observation of the preceding paragraph, it follows that $J(\alpha)$ with the order topology is homeomorphic to $(0, 1)$. Therefore the Long Line satisfies the locally euclidean portion of the definition for a topological 1-manifold. Since we also know this space is Hausdorff, it follows that the Long Line is indeed a topological 1-manifold.■

THEOREM. *The Long Line $\mathbb{LL}$ has the following properties:* (1) *It is connected.*

(2) *It is not Lindelöf.*

(3) *It is not second countable.*

(4) *It is not metrizable.*

(5) *It is not paracompact.*

(6) *It is not normal.*

Proof of (1)**:** The Long Line is the union of the open connected subsets $J(\alpha)$ for $\alpha > 1$; since 1 belongs to each of this set, it follows that their union, which is $\mathbb{LL}$, must also be connected.■

Proof of (2)**:** As in the previous paragraph, the Long Line is the union of the open subsets $J(\alpha)$ for $\alpha > 1$. We claim that no countable subcollection covers $\mathbb{LL}$. To see this, let $\{ J(\alpha_k) \}$ be a countable family. We then now that there is some $\beta \in S_\Omega$ such that $\beta > \alpha_k$ for all k. It then follows that that the union of the sets $J(\alpha_k)$ is contained in $J(\beta)$. Since the latter is a proper subset of $\mathbb{LL}$, it follows that the given countable subcollection cannot cover the Long Line. Therefore the latter cannot be Lindelöf.■

Proof of (3)**:** This follows from (2) because second countable spaces are always Lindelöf.■

Proof of (4)**:** Since connected metrizable topological manifolds are second countable and $\mathbb{LL}$ is a connected topological manifold that is not second countable, it follows that $\mathbb{LL}$ is not metrizable.■

Proof of (5)**:** Since paracompact topological manifolds are metrizable and $\mathbb{LL}$ is a topological manifold that is not metrizable, it follows that $\mathbb{LL}$ is not paracompact.■

Proof of (6)**:** Since paracompact Hausdorff spaces are normal and $\mathbb{LL}$ is Hausdorff but not paracompact, it follows that $\mathbb{LL}$ is not normal.■

EXAMPLES IN HIGHER DIMENSIONS

One can also construct examples of nonmetrizable topological n-manifolds with arbitrary dimension $n \geq 2$.

PROPOSITION. *For each $n \geq 2$ the cartesian product of n copies of $\mathbb{LL}$ with itself is a nonmetrizable topological n-manifold.*

Sketch of Proof: The product is a topological manifold because the product of n copies of a topological k-manifold with itself is always a topological nk-manifold (the product is Hausdorff and every point has an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^{nk}$). This product is not metrizable because an n-fold product of a topological space X with itself will be metrizable if and only if X itself is metrizable.∎

REFERENCES

[D] J. Dugundji, *Topology.* Allyn and Bacon, Boston, 1966.

[K] J. L. Kelley, *General Topology.* Van Nostrand, New York, 1955.