Math 205C
Spring 2005
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UPDATED GENERAL INFORMATION — JUNE 3, 2005

I shall be in my office between 3:00 and 4:30 P.M. on Monday, June 6.

About 40 per cent of the final examination will involve statements of definitions and results, while the remaining part will cover some basic consequences of the main definitions and results along with some relatively simple examples.

THINGS TO KNOW

These will be sorted by units.

I. Topological background

Definitions of topological manifolds, locally finite open coverings, partitions of unity. Universal mapping properties for products, subspaces, and quotient spaces.

II. Local theory of smooth functions

Definition of differentiability for continuous functions on open subsets of Euclidean spaces, description of the derivative matrix/linear transformation, the Chain Rule, smoothness of class $\mathcal{C}^r$. Statement of the Inverse Function Theorem, definitions of immersions and submersions, straightening lemmas for such maps, conditions when such maps are automatically open. Basic ideas behind the construction of bump functions and their use to construct smooth partitions of unity. Locally definitions of vector fields and integral flows, important properties of the integral flow mapping (where it is defined, retrieval of integral curves from it, continuity and smoothness properties, local 1-parameter groups of transformations).

III. Global theory of smooth manifolds and mappings

The basic examples of level sets for regular values of smooth functions and the fact that these are smooth submanifolds, definitions of smooth atlases, maximal atlases and smooth structures, constructions such as smooth structures on open subsets, products, covering spaces and their universal mapping properties, smooth approximations to continuous functions and the fact that sufficiently close approximations are homotopic to the original function, definitions of disjoint unions and their uses to construct spaces and manifolds out of pieces, construction of the tangent space and the latter's fundamental properties (vector bundle structure, smooth maps of manifolds yield maps of tangent spaces, basic formal relationships satisfied by such constructions), global definitions of immersion and submersion, generalizations of previous local results, equivalent formulations of the notion of smooth submanifold, existence of smooth embeddings into Euclidean spaces.

IV. Vector fields

Basic local and global definitions, constructing vector fields from local information, integral flows, completeness of vector fields and flows, basic properties of incomplete vector fields and flows, derivations on algebras, the defining conditions for a Lie algebra (anticommutative, Jacobi identity), Lie structure on the vector fields over a smooth manifold and the relation of vector fields to derivations on the algebra $\mathcal{C}^\infty(M)$, the significance of the equation $[X, Y] = 0$ for the 1-parameter groups of the vector fields X and Y.

V. Cotangent bundles, riemannian metrics and exterior forms

Construction of the cotangent bundle, definition of a differential 1-form on a manifold, riemannian metrics and their existence on every smooth manifold, local definition of differential forms, algebraic and differential constructions on forms and their basic identities, methods of extension to arbitrary smooth manifolds, relations between differential forms and classical vector analysis.

PROBLEMS TO WORK

Understanding these problems and solving them is very strongly recommended.

1. Let $O(3) \subset GL(3, \mathbf{R})$ be the subgroup of all orthogonal matrices (*i.e.*, the transpose is the same as the inverse). Prove that $O(3)$ is a smooth submanifold of $GL(3, \mathbf{R})$ by showing that 0 is a regular value for the smooth mapping into $\mathbf{R}^6$ whose coordinates are the real valued functions $|a_1|^2 - 1$, $|a_2|^2 - 1$, $|a_3|^2 - 1$, $\langle a_1, a_2\rangle$, $\langle a_1, a_3\rangle$ and $\langle a_2, a_3\rangle$.

Another example along the same lines would be the subgroup $SL(3, \mathbf{R})$ of all matrices with determinant 1; in this case the objective is to show that 0 is a regular value of the function $\det A - 1$.

2. One can construct the Klein bottle KB using two smooth charts (U_i, h_i) for $1 = 1, 2$ where $U_1 = U_2 = \mathbf{R}^2 - \{0\}$ such that the overlapping images are given by $V_{21} = V_{12} = \{\ z \mid |z| < \frac{1}{2}$ or $|z| > 2\ \}$ and the transition diffeomorphisms ψ_{ij} are both given by $\psi_{ij}(z) = |z|^{-1}z$ if $|z| > 2$ and $\psi_{ij}(z) = |z|^{-2}\overline{z}$ if $|z| < \frac{1}{2}$ (here we identify $\mathbf{R}^2$ with the complex numbers, and z may be viewed as a complex number). What are the domains of the charts for the corresponding smooth atlas of the tangent space $T(KB)$, and what is the corresponding transition map for these charts?
Another example upon which one could try this construction would be the Möbius strip.

3. A smooth manifold M of even dimension is said to be a *complex manifold* if there is a smooth atlas of charts (U_α, h_α) such that the derivatives of the transition maps $\psi_{\beta\alpha} =$ "$h_\beta^{-1}h_\alpha$" all lie in the image of the inclusion

$$GL(n, \mathbf{C}) \longrightarrow GL(2n, \mathbf{R})$$

given by interpreting each complex number $a + b\,i$ as the 2×2 matrix $aI + bJ$ where I is the 2×2 identity matrix and J is the 2×2 matrix defining counterclockwise rotation through a $90°$ angle. Prove that if P and Q are complex manifolds then so is their product $P \times Q$.

4. A smooth manifold M is said to be *orientable* if there is a smooth atlas of charts (U_α, h_α) such that the Jacobians of the transition maps $\psi_{\beta\alpha} =$ "$h_\beta^{-1}h_\alpha$" are all positive. Suppose that we are given such an atlas $\mathcal{A}$, and let $\mathcal{A}'$ consist of all smooth charts in the maximal atlas of the form $(V, h_\alpha|V)$ where (U_α, h_α) belongs to $\mathcal{A}$ and $V \subset U_\alpha$. Prove that $\mathcal{A}'$ also has the same Jacobian property. Using this, prove that if W is open in M and M is orientable, then W is also orientable.

5. A vector field is said to have compact support if it vanishes off some compact subset.
(i) Show that if X is a vector field with compact support and it vanishes off the compact set B, then the images of its integral curves are either one point sets or subsets of B.

(*ii*) Explain why (*i*) implies that X is complete.

6. Suppose that M and N are smooth manifolds such that $\dim M \geq \dim N$ such that M is compact and N is connected but NOT compact. Prove that there is no smooth map from M to N whose rank is the maximum possible value (namely, $\dim N$) at every point.

7. Suppose that ω and θ are closed differential forms on an open subset U and that $\omega - \omega'$ is exact.

(*i*) Why is θ' also closed?

(*ii*) Why is $\omega \wedge \theta$ also closed?

(*iii*) Why is $\omega \wedge \theta - \omega \wedge \theta'$ exact?

8. Let M be a smooth manifold, let τ_M^* be the projection for the cotangent bundle of M, and let $z : M \to T^*(M)$ be the zero section. Prove that the identity map of $T^*(M)$ is smoothly homotopic to the zero map $z \circ \tau_M^*$.

9. Let $g_{\mathbf{H}}$ be the riemannian metric on the upper half plane $\mathbf{H}$ (= all points with positive second coordinate) defined by the formula

$$g_{\mathbf{H}} = \frac{dx^2 + dy^2}{y^2} \ .$$

Find the gradient of $f(x) = xy$ with respect to this metric. [*Hint:* The definition of the gradient implies that $\nabla_g(f)$ is uniquely determined by the equation

$$g(\nabla_g(f), X) \quad = \quad df(X) \quad = \quad Xf$$

for all vector fields X. Write both X and $\nabla_g(f)$ as linear combinations of the unit vector fields

$$\frac{\partial}{\partial x} \ , \ \frac{\partial}{\partial y}$$

let X be one of these basic vector fields, and see what this implies for the coefficients of $\nabla_g(f)$.]

10. (*i*) Let ω be a nowhere zero smooth 1-form on a smooth compact manifold M. Show that if $\omega \wedge d\omega = 0$ then there exists a 1-form α such that $d\omega = \alpha \wedge \omega$ [*Hint:* First do it locally then use a partition of unity.]

(*ii*) Let ω be the 1-form on $\mathbf{R}^3$ defined by $x\,dy - y\,dx + dz$. Show that for every nonzero real valued function f the form $f\omega$ is not closed.