Nonsubmersion examples for compact manifolds

On page 129 of `lectnotes.pdf` we noted that regular homotopy classes of smooth submersions from a **noncompact** connected manifold M^m to another manifold N^n (where $m \geq n$) are in 1–1 correspondence with homotopy classes of continuous tangent bundle morphisms $F : T(M) \to T(N)$ satisfying the following conditions:

(i) There is a continuous map $f : M \to N$ such that $\tau_N \circ F = f \circ \tau_M$.

(ii) For each $x \in M$, the induced map F_x from $T_x(M)$ to $T_{f(x)}(N)$ is a linear surjection.

Furthermore, we mentioned that this classification does not extend to submersions defined on **compact** manifolds, and we described a class of pairs (F, f) such that properties (i) and (ii) hold but there is no smooth submersion $g : M \to N$ such that f is homotopic to g.

Our purpose here is to describe some similar, but simpler, examples for which we can give a self-contained proof that the underlying map $f : M \to N$ is not homotopic to a smooth submersion; in fact, in these examples f is not even homotopic to a **topological** submersion, and it is also possible to prove a homotopy-theoretic analog of these results involving a concept of **compact fiberings** (however, we shall not prove these generalizations).

Despite the file name for this note, our arguments do not require C. Ehresmann's theorem that a proper smooth submersion is a smooth fiber bundle projection (see the file `ehresmann.pdf` in the course directory). In fact, our aguments only use material at the level of this course and its two prerequisites.

The examples

For each positive integer k, let T^k denote the Cartesian product of k copies of the circle. Since the tangent bundle $\mathbf{T}(S^1)$ is isomorphic to the trivial vector bundle $S^1 \times \mathbf{R}^n$ and the tangent bundle of a product satisfies $\mathbf{T}(A \times B) \cong \mathbf{T}(A) \times \mathbf{T}(B)$, it follows that for each k we have $\mathbf{T}(T^k) \cong T^k \times \mathbf{R}^k$.

Suppose now that $m \geq n > 0$, and let $f : T^m \to T^n$ be the constant map. If $C : \mathbf{R}^m \to \mathbf{R}^n$ is an arbitrary linear transformation of maximum rank (for example, projection onto the first n coordinates), then one can form a map F satisfying properties (i) and (ii) with respect to f by taking the mapping

$$f \times C : T^m \times \mathbf{R}^m \longrightarrow T^n \times \mathbf{R}^n$$

and composing it with the isomorphisms

$$\mathbf{T}(T^m) \quad \cong \quad T^m \times \mathbf{R}^m \qquad\qquad \mathbf{T}(T^n) \quad \cong \quad T^n \times \mathbf{R}^n$$

described above.

Having shown the existence of a tangent bundle surjection F such that $\tau_N \circ F = f \circ \tau_M$, we shall have our examples if we can prove the following result.

Proposition. *If $m \geq n > 0$, then the constant map f from T^m to T^n is not homotopic to a smooth submersion.*

In the course of the proof we shall need the following elementary fact relating basepoint preserving homotopies and homotopies that are not necessarily basepoint preserving.

Lemma. *Let (X, x) be a topological space, let G be a connected topological group, and take the identity for the basepoint of G. Then two basepoint preserving maps from (X, x) to $(G, 1)$ are basepoint preservingly homotopic if and only if they are homotopic by a homotopy which is not necessarily basepoint preserving.*

This applies to our examples because every torus is a connected topological group.

Proof of the lemma. The $(\Rightarrow)$ implication is trivial, so it is only necessary to discuss the opposite implication. Let H be a homotopy which is not necessarily basepoint preserving, and define a closed continuous curve $\gamma(t) = H(x, t)$. Then $K(y, t) = H(y, t) \cdot \gamma(t)^{-1}$ defines a basepoint preserving homotopy relating the two original mappings.∎

Proof of the proposition. Suppose that f is homotopic to some smooth submersion, say g_0. Our first step will be to show that g_0 is homotopic to a basepoint preserving smooth submersion. This is fairly easy to do. Since T^k is arcwise connected, there is a continuous curve α joining 1 to $g_0(1)$, and since the group operation on T^k is smooth it follows that translation by a group element is a diffeomorphism; therefore the map $g_0(z) \cdot \alpha(t)^{-1}$ defines a regular homotopy from g_0 to a smooth submersion which is basepoint preserving. Let g denote this basepoint preserving submersion.

By the lemma we know that g is basepoint preservingly homotopic to the constant map, and therefore the associated map of fundamental groups

$$g_* : \pi_1(T^m) \to \pi_1(T^n)$$

is trivial. Therefore, the lifting criterion implies the existence of a continuous mapping $h : T^m \to \mathbf{R}^n$ such that $g = p \circ h$, where $p : \mathbf{R}^n \to T^n$ is the usual simply connected covering space projection. In fact, since the covering space projection p is a smooth submersion, it follows that h must also be a smooth submersion. Since smooth submersions are open mappings, it follows that the image of h must be **open** in $\mathbf{R}^n$.

On the other hand, since T^m is compact, it follows that the image of h must be a **compact** subset of $\mathbf{R}^n$. This is impossible because $\mathbf{R}^n$ has no nonempty subsets that are both compact and open. The source of the contradiction is our assumption that the constant map is homotopic to a smooth submersion, and therefore no such mapping can exist, which is exactly what we wanted to prove.∎