Proper Maps and Universally Closed Maps

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Continuous functions defined on compact spaces generally have special properties. For example, if X is a compact space and Y is a Hausdorff space, then f is a closed mapping. There is a useful class of continuous mappings called *proper*, *perfect*, or *compact* maps that satisfy many properties of continuous maps with compact domains. In this note we shall study a few basic properties of these maps. Further information about proper maps can be found in Bourbaki [B$_2$], Section I.10), Dugundji [D] (Sections XI.5–6, pages 235–240), and Kasriel [K] (Sections 95 and 105, pages 214–217 and 243–247). There is a slight difference between the notions of perfect and compact mappings in [B$_2$], [D], and [K] and the notion of proper map considered here. Specifically, the definitions in [D] and [K] include a requirement that the maps in question be surjective. The definition in [B$_2$] is entirely different and corresponds to the notion of *universally closed* map discussed later in these notes; the equivalence of this definition with ours for a reasonable class of spaces is proved both in [B$_2$] and in these notes (see the section *Universally closed maps*).

PROPER MAPS

If $f : X \to Y$ is a continuous map of topological spaces, then f is said to be *proper* if for each compact subset $K \subset Y$ the inverse image $f^{-1}(K)$ is also compact.

EXAMPLE. *Suppose that A is a finite subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n - A \to \mathbb{R}^m$ is a continuous function such that for all $a \in A$ we have*

$$\lim_{x \to a} |f(x)| = \lim_{x \to \infty} |f(x)| = \infty.$$

Then f is proper.

Proof. If $K \subset \mathbb{R}^m$ is compact, then K is contained in some large disk D. The limit condition at ∞ implies that $f^{-1}(D)$ is contained in some large disk $D' \subset R^n$; furthermore the limit conditions at all the points $a_i \in A$ imply that $f^{-1}(D)$ is also contained in the complement of $E = D' - \cup\, N_i$, where N_i is a small open disk neighborhood with center a_i. But E is compact and $f^{-1}(K)$ is a closed subset of $\mathbb{R}^n$ that is contained in E; it follows that $f^{-1}(K)$ is compact.■

IMPORTANT SPECIAL CASE. *Suppose $n = m = 2$ and $f(z) = p(z)/q(z)$, where p and q are complex polynomials with deg $p >$ deg q. Then f satisfies the limit condition and therefore is proper.*

The first result, which is fairly standard, shows that proper maps of certain not necessarily compact spaces have some important properties in common with continuous maps of compact Hausdorff spaces.

THEOREM 1. *Suppose that X and Y are Hausdorff spaces with Y either locally compact or metrizable, and let $f : X \to Y$ be continuous. Then f is proper if and only if f is closed and for each $y \in Y$ the set $f^{-1}(\{y\})$ is compact.*

Proof: Suppose that f is closed and inverse images of one point sets are compact; we claim that inverse images of arbitrary compact subsets are compact.

Let K be a compact subset of Y, and let $\mathcal{F} = \{F_\alpha\}$ be a collection of closed subsets of $f^{-1}(K)$ with the finite intersection property. Without loss of generality, we may assume that $\mathcal{F}$ is closed under finite intersections, for if $\mathcal{F}'$ is the family of all finite intersections of sets in $\mathcal{F}$ then $\mathcal{F}'$ is closed under finite intersections and the intersection of all the sets in $\mathcal{F}$ equals the intersection of all the sets in $\mathcal{F}'$. Since f is closed, the subsets $f(F_\alpha)$ are closed in Y; since $F_\alpha \subset f^{-1}(K)$ is true by assumption, it follows that $f(F_\alpha) \subset K$. Combining these observations, we see that $f(F_\alpha)$ is a compact subset of K. But the family of closed sets $f(F_\alpha)$ also has the finite intersection property because

$$\varnothing \neq f(F_{\alpha_1} \ \cap \ \cdots \ \cap \ f(F_{\alpha_k}) \subset f(F_{\alpha_1}) \ \cap \ \cdots \ \cap \ f(F_{\alpha_k}).$$

Therefore $\varnothing \neq \cap f(F_\alpha)$ by compactness of K. Let y be a point in the intersection. Since the family $\mathcal{F}$ is closed under finite intersections, for all $\alpha_1, \ \cdots \ , \alpha_n$ we have $f^{-1}(\{y\} \cap F_{\alpha_1} \cap \cdots \cap F_{\alpha_n} \neq \varnothing$. Therefore the family $\{ f^{-1}(\{y\}) \ \cap \ F_\alpha\}$ has the finite intersection property. But $f^{-1}(\{y\})$ is compact, and therefore $\varnothing \neq f^{-1}(\{y\}) \ \cap \ (\cap_\alpha \ F_\alpha) \subset \cap_\alpha \ F_\alpha$. Therefore the set $f^{-1}(K)$ is compact. $\square$

Suppose now that f is proper and Y is locally compact or metrizable. If $F \subset X$ is a closed subset, then it is immediate that $f|F$ is proper. Therefore it suffices to prove that if f is proper then $f(X)$ is closed in Y (because $f(F) = f|F\,(F)$).

Assume first that Y is locally compact. Let y be a point in the closure of $f(X)$, and let K be a compact neighborhood of y. Then $f^{-1}(K)$ is a nonempty compact set, and $f|\, f^{-1}(K)$ a closed mapping. Therefore $f(\, f^{-1}(K)\,) = f(X) \ \cap \ K$ is a closed set. But by construction y is a limit point of this set, and consequently $y \in f(X) \ \cap K \ \subset f(X)$. $\square$

Assume now that Y is a metric space. Let F be closed in X, and let $\{x_n\}$ be a sequence of points in F such that $\lim f(x_n) = y$. Let C_n be the compact set $\{y, f(x_n), f(x_{n+1}), ...\}$, so that $f^{-1}(C_n)$ is a nonempty nested sequence of compact sets. Therefore by compactness we have that $\varnothing \neq \ \cap_n \ (\, f^{-1}(C_n) \ \cap F\,) \ = \ (\, f^{-1}(\cap_n \ C_n)\,) \ \cap \ F = f^{-1}(\{y\}) \ \cap F$. Therefore $y \in f(F)$ must hold, and consequently $f(F)$ must be closed in Y.$\blacksquare$

Remark, The ($\Longleftarrow$) implication does not require an extra condition on Y. The ($\Longrightarrow$) implication is valid more generally if Y is a k-space (see [D], Section XI.9, especially XI.9.3 on p. 248). Here is the proof that f is closed under this hypothesis: The set $f(X)$ is closed if and only if $f(X) \cap K$ is compact for all compact subsets $K \subset Y$. But $f(X) \cap K \ = \ f(\, f^{-1}(K)\,)$, and this set is compact because f proper $\Longrightarrow f^{-1}(K)$ is compact and $f^{-1}(K)$ compact $\Longrightarrow f(\, f^{-1}(K)\,)$ compact. $\square$

THEOREM 2. *Suppose that $f : X \to Y$ is a proper map and $B \subset Y$. Then $f|f^{-1}(B)$ is proper. Conversely, if $\{B_\alpha\}$ is either an open covering or a finite closed covering of Y and each of the maps $f|f^{-1}(B_\alpha)$ is proper, then f is also proper.*

Proof. ($\Longrightarrow$) Let $f_B = f|f^{-1}(B)$. If K is a compact subset of B then $f_B^{-1}(K) = f^{-1}(K)$; but the latter is compact since f is proper, and therefore it follows that f_B is proper.

($\Longleftarrow$) Let $f_\alpha = f|f^{-1}(B_\alpha)$, and let $K \subset Y$ be compact. Suppose that $\{B_\alpha\}$ is an open covering of Y. Then by compactness K is contained in a finite union $B_{\alpha_1}, \cup \cdots \cup B_{\alpha_k}$. Let $V_{\alpha_j} = K \ \cap \ B_{\alpha_j}$; then the sets $\{V_{\alpha_j}\}$ define a finite open covering of K, and by Munkres, Theorem 36.1, pp. 225–226 (especially Step 1), we can find closed (hence compact) subsets $F_{\alpha_j} \subset V_{\alpha_j}$ such that $K = \cup_j F_{\alpha_j}$. Therefore it follows that $f^{-1}(K) = \cup_j \ f_{\alpha_j}^{-1}(F_{\alpha_j})$. But each map $f_{\alpha_j}^{-1}$ is proper, and hence it follows that each subset of the form $f_{\alpha_j}^{-1}(F_{\alpha_j})$ is compact. Since a finite union of compact sets is compact, it follows that $f^{-1}(K)$ is compact.

Now suppose that $\{B_\alpha\}$ is a finite closed covering. Let $K_\alpha = K \ \cap \ B_\alpha$; then each set K_α is compact because each set B_α is closed in Y. Since each map f_α is proper we know that each set $f_\alpha^{-1}(K_\alpha) = f^{-1}(K_\alpha)$ is compact. But $f^{-1}(K) = \cup_\alpha \ f_\alpha^{-1}(K_\alpha)$, and therefore $f^{-1}(K)$ is compact because a finite union of compact sets is compact.■

Remark. The theorem also holds for *locally finite* closed coverings.

THEOREM 3. —it Let $f_\alpha : X_\alpha \to Y_\alpha$ be continuous for each $\alpha \in J$, and let

$$\prod f_\alpha : \prod X_\alpha \to \prod Y_\alpha$$

be the associated map of product spaces. Then $\prod f_\alpha$ is proper if and only if each f_α is proper.

Proof. ($\Longrightarrow$) Let $\beta \in J$ be arbitrary, and for each $\alpha \neq \beta$ choose $x_\alpha \in X_\alpha$. Furthermore, for each $\alpha \neq \beta$ let $y_\alpha = f_\alpha(x_\alpha)$. Let $i_\beta : X_\beta \to X_\beta \times \prod \alpha \neq \beta\{x_\alpha\}$ be the identity on the β factor and constant on the other factors, and let $j_\beta : Y_\beta \to Y_\beta \times \prod \alpha \neq \beta\{y_\alpha\}$ be the identity on the β factor and constant on the other factors; then i_β and j_β are closed subspace inclusions and

$$(\prod f_\alpha) i_\beta = j_\beta f_\beta.$$

Let $K \subset Y_\beta$ be compact. Then the formula above implies that

$$i_\beta(f_\beta^{-1}(K)\,) = (\prod f_\alpha)^{-1}(j_\beta(K))$$

and since the product map is proper it follows that $i_\beta(f_\beta^{-1}(K)\,)$ is compact. But i_β is a homeomorphism onto a closed subset, and therefore a subset $B \subset X_\beta$ is compact if and only if $i_\beta(B)$ is compact. Therefore it follows that $f_\beta^{-1}(K)$ is compact. $\square$

($\Longleftarrow$) Suppose that K is compact in $\prod Y_\alpha$. Then $L_\alpha = \pi_\alpha(L)$ is a compact subset of Y_α. By Tychonoff's Theorem the product $\prod L_\alpha$ is compact. It is immediate that $(\prod f_\alpha)^{-1}(K)$ is a closed subset of $\prod f_\alpha^{-1}(L_\alpha)$. But the factors $f_\alpha^{-1}(L_\alpha)$ are all compact because the maps f_α are proper, and therefore the product is compact by Tychonoff's Theorem. The observations of the preceding two sentences combine to show that $(\prod f_\alpha)^{-1}(K)$ is compact.■

THEOREM 4. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous.*
(i) If f and g are proper, then gf is also proper.
(ii) If gf is proper, then f is proper.

(*iii*) *If gf is proper and f is onto, then g is proper.*

Proof. (*i*) Let K be a compact subset of Z. Then $g^{-1}(K)$ is a compact subset of Y because g is proper; but f is also proper, and therefore $(gf)^{-1}(K) = f^{-1}g^{-1}(K)$ is also compact. $\square$

(*ii*) Let C be a compact subset of Y. Then $g(C)$ is compact, and therefore $f^{-1}g^{-1}(g(C))$ is a compact subset of X. Since $C \subset g^{-1}(g(C))$ and $f^{-1}(C)$ is a closed subset of X, it follows that $f^{-1}(C)$ is a closed subset of the compact set $f^{-1}g^{-1}(g(C))$, and therefore $f^{-1}(C)$ is compact. $\square$

(*iii*) Let K be a compact subset of Z. Since gf is proper the set $(gf)^{-1}(Z) = f^{-1}g^{-1}(Z)$ is compact. However, f is onto and therefore $B = f(f^{-1}(B))$ for all $B \subset Y$. It follows that $g^{-1}(K) = f(f^{-1}g^{-1}(K))$, and the set on the right hand side is compact because it is the image of the compact set $(gf)^{-1}(Z) = f^{-1}g^{-1}(Z)$ under the continuous function f.$\blacksquare$

UNIVERSALLY CLOSED MAPS

Proper maps arise naturally in many analytic and geometric contexts (compare [B_1] and [Se]). For these purposes it is convenient to have an alternate characterization of proper maps; in fact, the definition in [B_2] takes this characterization as the definition of proper map. Since it is difficult to find an account of this outside of [B_2], the details are given below.

Definition. Let $\mathcal{A}$ be a family of Hausdorff spaces such that the following hold:
(*i*) If $A \in \mathcal{A}$ and B is homeomorphic to A then $B \in \mathcal{A}$.
(*ii*) $\mathcal{A}$ contains all one point spaces.

If $X, Y \in \mathcal{A}$, then a continuous map $f : X \to Y$ is said to be *universally closed* (with respect to $\mathcal{A}$) if for each $Z \in \mathcal{A}$ the map $f \times 1_Z : X \times Z \to Y \times Z$ is a closed mapping. Notice that if $\mathcal{B} \subset \mathcal{A}$ and f is $\mathcal{A}$-universally closed, then f is also $\mathcal{B}$-universally closed.

A space $X \in \mathcal{A}$ is said to be $\mathcal{A}$-*complete* if the constant map $X \to \{\text{pt.}\}$ is proper.

LEMMA 5. *The composite of two $\mathcal{A}$-universally closed maps is $\mathcal{A}$-universally closed.*

The proof of this is elementary.$\blacksquare$

THEOREM 6. *Let $\mathcal{A}$ be one of the following families:*
(*i*) *All $T_{3\frac{1}{2}}$ spaces.*
(*ii*) *All locally compact Hausdorff spaces.*
(*iii*) *All closed subsets of $\mathbb{R}^n$, where n ranges over all nonnegative integers.*

Then a space $X \in \mathcal{A}$ is $\mathcal{A}$-complete if and only if X is compact.

Proof. To prove the ($\Longleftarrow$) implication, it suffices to show that if X is compact Hausdorff and Y is $T_{3\frac{1}{2}}$ then the second coordinate projection $\pi_Y : X \times Y \to Y$ is a closed mapping.

Let F be a closed subset of $X \times Y$, and let $W = X \times Y - F$; we claim that W is open. If $y \notin \pi_Y(F)$, then for every $x \in X$ there is an open subset $U_x \times V_x \subset X \times Y$ such that $x \in U_x$, $y \in V_x$, and $U_x \times V_x \ \cap \ F \ = \ \varnothing$. The family $\{U_x\}$ forms an open covering of $X \times \{y\}$, and by compactness of X there is a finite subcovering $U_{x_1}, \cdots U_{x_k}$. Let $V = \cap V_{x_i}$. Then $\cup (U_{x_i} \times V) \cap F = \varnothing$, and consequently $V \cap \pi_Y(F) = \varnothing$, so that $y \in V$. □

The proof of the ($\Longrightarrow$) implication separates into three cases depending on $\mathcal{A}$.

Case 1. Suppose that $\mathcal{A}$ is all $T_{3.5}$ spaces. If X is $T_{3.5}$ but not compact, then X is homeomorphic to a proper subset of its Stone-Čech compactification. Let $F \subset X \times \beta X$ be the graph of the embedding of X in βX. Then F is closed in $X \times \beta X$, but the projection $\pi_{\beta X}(X) \subset \beta X$ is a dense proper subset. □

Case 2. Suppose that $\mathcal{A}$ is all locally compact Hausdorff spaces. The proof in Case 1 goes through if one replaces the Stone-Čech compactification βX with the one point compactification $X^\bullet$. □

Case 3. Suppose that $\mathcal{A}$ is all closed subsets of $\mathbb{R}^n$, where n ranges over all nonnegative integers. If X is a noncompact closed subset of $\mathbb{R}^k$ then its one point compactification $X^\bullet$ is a closed subspace of $(\mathbb{R}^k)^\bullet \approx S^n \subset \mathbb{R}^{k+1}$. It follows that $X^\bullet \in \mathcal{A}$, and this implies that the argument in Case 1 can be modified to apply here, with $X^\bullet$ repacing βX.■

THEOREM 7. *Let $\mathcal{A}$ be either the set of locally compact Hausdorff spaces or the set of closed subspaces of the spaces $\mathbb{R}^k$. Then a continuous map $f : X \to Y$ of spaces in $\mathcal{A}$ is $\mathcal{A}$-universally closed if and only if it is proper.*

Remark. If $\mathcal{A}$ is the class of all Hausdorff spaces, then there are proper maps $f : X \to Y$ such that $X, Y \in \mathcal{A}$ but f is not universally closed. One example is indicated in [B_2], Exc. I.10.4: Given an uncountable family of topological spaces X_α, define a modified product topology $\prod'$ with subbase given by all products of open sets $U_\alpha \subset X_\alpha$ where $U_\alpha = X_\alpha$ for all but at most countably many α. If we take each X_α to be a two point space with the discrete topology, then it is immediate that $Y = \prod' X_\alpha$ is $\mathbf{T_3}$ and that the "identity" map j from $X = \prod X_\alpha$ to Y is continuous. Since j is not a homeomorphism but j is continuous, 1–1, and onto, it follows that j cannot be closed. On the other hand, it is known that every compact subset of Y is *finite* (compare [B_2], Exc. I.9.4), and therefore j is proper (in the sense of the definition in these notes).

Proof. ($\Longrightarrow$) If f is $\mathcal{A}$-universally closed and $K \subset Y$ is compact, then $K \to \{pt.\}$ is universally closed (Theorem 5), and the map $f_K : f^{-1}(K) \to K$ given by $f|f^{-1}(K)$ is also universally closed (a routine verification). Since the composite of universally closed maps is universally closed (see Lemma 5), it follows that the constant map $f^{-1}(K) \to \{\text{pt.}\}$ is also universally closed. But by Theorem 5 this implies that $f^{-1}(K)$ is compact. □

($\Longleftarrow$) If f is proper, then by Theorem 3 we know that $f \times 1_Z$ is proper for all spaces Z (because an identity map is always proper). Therefore by Theorem 1 the map $f \times 1_Z$ is closed.■

When working with $\mathcal{A}$-universally closed maps it is often useful to have a smaller set of test spaces $\mathcal{A}_0$ such that a map is universally closed if and only if its product with the identity map of a space in $\mathcal{A}_0$ is closed. Here is an abstract version and its most important special case(s).

LEMMA 8. *Let $\mathcal{A}$ be a family of Hausdorff spaces satisfying the conditions at the beginning of this section, and let $\mathcal{A}_0$ be a subfamily of $\mathcal{A}$ such that every space in $\mathcal{A}$ is homeomorphic to a* *closed subset of a space in $\mathcal{A}_0$. Let X and Y be spaces in $\mathcal{A}$. Then a continuous map $f : X \to Y$ is $\mathcal{A}$-universally closed if and only if $f \times 1_E$ is closed for all spaces $E \in \mathcal{A}_0$.*

Important note: It is not assumed that X or Y lies in the subfamily $\mathcal{A}_0$.

Proof. The ($\Longleftarrow$) implication is immediate. To prove the ($\Longrightarrow$) implication, let $Z \in \mathcal{A}$, and let Z be homeomorphic to a closed subspace of E_0. Without loss of generality we can assume that Z is in fact a subspace of E_0. Then $X \times Z$ is a closed subset of $X \times E_0$, and therefore it follows that $f \times 1_{E_0}|X \times Z$ is a closed mapping. But this map factors as $j(f \times 1_Z)$, where j is the closed map defined by the inclusion $X \times Z \subset X \times E_0$. It follows that $f \times 1_Z$ must be a closed map.■

Examples. Suppose that $\mathcal{A}$ is the family of all closed subsets of $\mathbb{R}^n$, where n ranges over all positive integers. Then two choices for $\mathcal{A}_0$ are the family $\mathcal{E}_{\mathbb{R}}$ consisting of all the spaces $\mathbb{R}^n$ and the family $\mathcal{E}_{\mathbb{C}}$ consisting of all the spaces $\mathbb{C}^n$.

THEOREM 9. *Let $\mathcal{A}$ be the family of all closed subsets of $\mathbb{R}^n$, where n ranges over all positive integers, let $X, Y \in \mathcal{A}$, and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then a continuous map $f : X \to Y$ is proper if and only if $f \times 1_{\mathbb{F}^n}$ is closed for all positive integers n.*

Proof. By Theorem 6 the map f is proper if and only if it is $\mathcal{A}$-universally closed, and by Lemma 8 this is true if and only if $f \times 1_E$ is closed for all $E \in \mathcal{E}_{\mathbb{R}}$ or $\mathcal{E}_{\mathbb{C}}$.■

In algebraic and complex analytic geometry this result is used extensively (compare [B_1]).

EXERCISES

1. Suppose that $f : X \to Y$ is a continuous map of noncompact locally compact T_2 spaces. Let $f^\bullet : X^\bullet \to Y^\bullet$ be the map of one point comapactifications defined by $f^\bullet|X = f$ and $f^\bullet(\infty_X) = (\infty_Y)$. Prove that f is proper if and only if $f^\bullet$ is continuous.

2. Prove analogs of Theorems 6 and 7 for $\mathcal{A}$ = the class of all separable metric spaces. [*Hint:* Why is every separable metric space homeomorphic to a subspace of a compact one?]

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