SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205C

SPRING 2005

I. Topological Background

I.1 : Topological manifolds

Additional exercise

1. Let X be a topological space. Prove that X is a topological 0-manifold if and only if X is discrete.

SOLUTION.

If X is discrete and $x \in X$, then $\{x\}$ is an open neighborhood of x that is homeomorphic to the open set $\{0\}$, which is all of $\mathbf{R}^0$. Conversely, if X is a topological 0-manifold and $x \in X$, let U be an open subset containing x that is homeomorphic to an open subset of $\mathbf{R}^0$. Since the only nonempty open subsets of $\mathbf{R}^0$ has one point, this means that U must consist of x and nothing else. Therefore the one point set $\{x\}$ is open. Since x was arbitrary this means that X is discrete.■

2. Suppose that X is a topological manifold and U is an open subset of X. Prove that U is a topological manifold.

SOLUTION.

Let $x \in U \subset X$. Then there is an open subset $v \subset X$ such that $x \in V$ and V is homeomorphic to an open subset of $\mathbf{R}^n$. Since $U \cap V$ is open in V it follows that $U \cap V$ is also homeomorphic to an open subset of $\mathbf{R}^n$. Moreover, U is Hausdorff because a subset of a Hausdorff space is always Hausdorff. Therefore U is a topological manifold.■

3. Let X be a Hausdorff space, and suppose that X has an open covering $\{U_\alpha\}$ such that each U_α is a topological n-manifold for some fixed n. Prove that X is a topological manifold. Give a counterexample to this statement if the Hausdorff condition is removed.

SOLUTION.

Suppose that $x \in X$, and choose U_α such that $x \in U_\alpha$. Then there is an open subset $V \subset U_alpha$ such that $x \in V$ and V is homeomorphic to an open subset of $\mathbf{R}^n$. Since V is open in U_α and the latter is open in $\mathbf{R}^n$, it follows that V is an open neighborhood of x in X. Therefore X is a topological n-manifold.

The Forked Line F is an example of a space that is not Hausdorff but has an open covering consisting of topological manifolds. Recall that the latter is a quotient space of $\mathbf{R} \times \{0,1\}$; we claim that the image of each subset $\mathbf{R} \times \{j\}$ is open in the quotient space F and that this image is homeomorphic to $\mathbf{R}$. — We shall first verify the openness assertion. By the definition of the quotient topology, the openness of this image is equivalent to the openness of the set of all points in $\mathbf{R} \times \{0,1\}$ that are equivalent to points of $\mathbf{R} \times \{j\}$. The set of all such points is equal all points except $(0, 1-t)$, and this complement is indeed open in $\mathbf{R} \times \{0,1\}$. — Next, we show that the map(s) from $\mathbf{R} \times \{j\}$ to F are homeomorphisms onto their image. The quickest way to do this is to note that the projection map

$$\mathbf{R} \times \{0,1\} \longrightarrow \mathbf{R}$$

passes to a map on the quotient spaces $\widetilde{q} : F \to \mathbf{R}$. If

$$\alpha_j : \mathbf{R} \times \{j\} \longrightarrow F$$

is the inclusion map then the inverse to the associated map from $\mathbf{R} \times \{0,1\}$ to Image(α_j) is given by the restriction of $\widetilde{q}$ to Image(α_j)■

4. (i) Suppose that X is a topological n-manifold and Y is a topological m-manifold. Prove that $X \times Y$ with the product topology is a topological $(m+n)$-manifold.

SOLUTION.

First of all $X \times Y$ is Hausdorff because a product of Hausdorff spaces is Hausdorff. Let $(x,y) \in X \times Y$. Then there are open neighborhoods U and V of x and y respectively (in X and Y respectively) such that U is homeomorphic to an open subset of $\mathbf{R}^n$ and V is homeomorphic to an open subset of $\mathbf{R}^m$. It follows that $U \times V$ is homeomorphic to an open subset of $\mathbf{R}^n \times \mathbf{R}^m \cong \mathbf{R}^{m+n}$ and it contains (x,y). This proves that $X \times Y$ is a is a topological $(m+n)$-manifold.■

(ii) Suppose that E and X are connected Hausdorff spaces and $p : E \to X$ is a covering space projection. Prove that E is a topological n-manifold if and only if X is.

SOLUTION.

Suppose first that B is a topological n-manifold. Let $x \in E$, so that $p(x) \in B$. Let U be an evenly covered neighborhood of $p(x)$ in B. Then we know that there is an open subset $V \subset U$ such that $x \in V$ and V is homeomorphic to an open subset of $\mathbf{R}^n$. Now $p^{-1}(V)$ is a union of pairwise disjoint open subsets W_α such that p maps each subset homeomorphically onto V. Choose β so that $x \in W_\beta$. Then in particular W_β is homeomorphic to V and hence to an open subset of $\mathbf{R}^n$. Since E is Hausdorff, it must be a topological manifold.

Conversely, suppose now that E is a topological manifold. Let $x \in B$, and let U be an evenly covered open neighborhood of x. Choose $y \in E$ so that $p(y) = x$, and let U_0 be an open neighborhood of y in E such that p maps U_0 homeomorphically onto an open neighborhood of x. Since E is a topological manifold there is a subneighborhood $W_0 \subset U_0$ such that $y \in W_0$ and W_0 is homeomorphic to an open subset in $\mathbf{R}^n$. Then p will map W_0 homeomorphically onto an open subset W of B such that $x \in W$. It follows that W is an open neighborhood of x that is homeomorphic to an open subset of $\mathbf{R}^n$. By assumption we know that B is Hausdorff and therefore it follows that B is a topological manifold.■

5. THIS IS A REPETITION OF THE PREVIOUS PROBLEM 3 AND HAS THEREFORE BEEN DELEDTED.

6. A compact Hausdorff space Γ is a *graph* if it is a finite union of subspaces E_j such that each E_j is homeomorphic to the closed unit interval $[0,1]$ and if $i \neq j$ then $E_i \cap E_j$ is an endpoint of both E_i and E_j (note that one can characterize the endpoints topologically as the two points whose complements are connected). The set of endpoints of the subsets E_k is called the set of vertices of Γ and each E_k is called an edge of Γ. Prove that if Γ is a topological manifold, then every vertex lies on exactly two edges. [*Hint:* Look at the proof that the figure 8 curve is not a topological manifold.]

SOLUTION.

Every vertex lies on finitely many edges of the graph. Our first task is to show how one can retrive the number of edges containing a vertex from purely topological considerations.

Given a topological space X, let $\mathbf{cc}(X)$ denote the set of connected components of X. If $f : X \to Y$ is continuous then f takes all points in a single connected component of X to a single connected component of Y and therefore we have a well defined mapping of sets $f_* : \mathbf{cc}(X) \to \mathbf{cc}(Y)$. Observe that if f is the identity map on a space then f_* is also the identity, and if $g : Y \to Z$ is another continuous map then $(g \circ f)_* = g_* \circ f_*$; each of these follows by direct calculation (*cf.* the corresponding arguments for fundamental groups, where one has a homomorphism associated to a continuous map of pointed spaces).

We need these associated mappings to formulate the following basic topological property of graphs:

(1) **Topological characterization of the number of edges containing a given vertex.** *Let Γ be a graph, let v be a vertex fo Γ, and let n be the number of edges of the graph containing v. Then there is a neighborhood base $\{ U_\alpha \}$ of v such that $\mathbf{cc}(U_\alpha - \{v\})$ has n elements for all α and if $U_\beta \subset U_\alpha$, then the inclusion $j : U_\beta - \{v\} \to U_\alpha - \{v\}$ defines an isomorphism of sets.*

Since this statement is about a neighborhood base, it suffices to show that it is true if Γ is replaced by an arbitrary open neighborhood W of v (a neighborhood base for v in W is a neighborhood base for v in Γ). If F is the union of all edges that do not contain v as a vertex, then F is closed in Γ and its complement is a union of subspaces E'_j homeomorphic to $[0, 1)$ such that v corresponds to 0 in each case and $\{v\}$ is the intersection of any two such subspaces. The desired neighborhood base is then the union of the sets $E'_j(n)$ which correspond to $[0, \frac{1}{n})$ under the given homeomorphisms.■

Suppose now that X is a topological 1-manifold. Then we have the following somewhat similar result which is implicit in the hint for the problem:

(2) *Let X be a topological 1-manifold, and let $x \in X$. Then there is a neighborhood base $\{ W_\gamma \}$ of x such that $\mathbf{cc}(W_\gamma - \{x\})$ has 2 elements for all α and if $W_\delta \subset W_\gamma$, then the inclusion $j : W_\delta - \{x\} \to W_\gamma - \{x\}$ defines an isomorphism of sets.*

As in the previous case, it suffices to prove the result for an arbitrary neighborhood of a point, and since each point has a neighborhood homeomorphic to $\mathbf{R}$ we might as well assume that $X = \mathbf{R}$. In this case the desired neighborhood base of a point a is given by the open intervals $(a - \frac{1}{n},\ a + \frac{1}{n})$.■

Suppose now that Γ is a topological 1-manifold, and let v be a vertex of Γ that lies on n edges. Then there is a neighborhood base $\{ U_\alpha \}$ as described in (1) above. Likewise, since Γ is a 1-manifold there is a neighborhood base $\{ W_\gamma \}$ as described in (2) above. Given some U_α we can find W_γ and U_β such that $U_\beta \subset W_\gamma \subset U_\alpha$. Consider now the associated maps of connected components for the complements of $\{v\}$:

$$\mathbf{cc}(U_\beta - \{v\}) \longrightarrow \mathbf{cc}(W_\gamma - \{v\}) \longrightarrow \mathbf{cc}(U_\alpha - \{v\})$$

By the first result, the composite is an isomorphism from one set with n elements to another with n elements. Elementary considerations then imply that the map from $\mathbf{cc}(U_\beta - \{v\})$ to $\mathbf{cc}(W_\gamma - \{v\})$ must be 1–1 (if not the composite would not be 1–1 either). Since $\mathbf{cc}(W_\gamma - \{v\})$ has two elements this forces the inequality $n \leq 2$.

To complete the argument we only need to show that $n = 1$ is also impossible. But in this case there is a neighborhood base U_α of v such that each of the sets $U_\alpha - \{v\}$ is connected, and this does not happen in a 1-manifold. Therefore the only possibility is $n = 2$, and this completes the entire argument.■

7. In the notation of the previous exercise, it follows that if one removes a finite set of points from Γ, then Γ is a topological 1-manifold. All letters in the alphabet and all Hindu-Arabic numerals admit decompositions into closed subspaces which make them into graphs. Assuming that the letters and numerals are given in the sans-serif form

A B C D E F G H I J K L M

N O P Q R S T U V W X Y Z

1 2 3 4 5 6 7 8 9 0

determine the least numbers of points that must be removed in order to obtain a topological manifold.

SOLUTION.

The first thing to understand is exactly which points must be removed. If we delete all vertices then we have a space that is a pairwise disjoint union of finitely open subsets, each of which is homeomorphic to $(0, 1)$ and hence is an open interval. If we now add back all vertices that lie on exactly two edges, then we still have a topological manifold, for these points also have open neighborhoods that are homeomorphic to open intervals (consider the special neighborhood for v that was constructed in the first paragraph of the proof for assertion (1) in the previous exercise). On the other hand, by the preceding exercise we must remove all vertices that lie on n edges for each $n \neq 2$. Let ν_k be the number of vertices that lie on exactly k edges. In the table below we list the values of ν_k for $k \neq 2$; these are simply given by visual inspection of the letters and numberals as they are depicted in the problem (sans-serif characters were used to simplify everything; the answers would be different if we used ordinary typset letters and numerals). For the characters we display, one always has $\nu_k = 0$ for $k \geq 5$; if we carried out a similar exercises for the Cyrillic alphabet there would be an example (the character representing "zh") for which $\nu_6 \neq 0$. The final column in the table adds the numbers ν_k for $k \neq 2$, and this is the number of points that must be removed in order to obtain a topological 1-manifold.

(see the next page)

Numbers of vertices on k edges ($k \neq 2$)

SYMBOL	ν_1	ν_3	ν_4	TOTAL
A	2	2	0	4
B	0	2	0	2
C	2	0	0	2
D	0	0	0	0
E	3	1	0	4
F	3	1	0	4
G	2	0	0	2
H	4	2	0	6
I	2	0	0	2
J	2	0	0	2
K	4	2	0	4
L	2	0	0	2
M	2	0	0	2
N	2	0	0	2
O	0	0	0	0
P	1	1	0	2
Q	2	0	1	3
R	2	2	0	4
S	2	0	0	2
T	3	1	0	3
U	2	0	0	2
V	2	0	0	2
W	2	0	0	2
X	4	0	1	5
Y	3	1	0	4
Z	2	0	0	2
1	3	1	0	4
2	2	0	0	2
3	2	0	0	2
4	2	0	1	3
5	2	0	0	2
6	1	1	0	2
7	2	0	0	2
8	0	0	1	1
9	1	1	0	2
0	0	0	0	0

One can obviously ask similar questions for other familiar characters like #, $, %, & or $*$, and if this is not enough there is always the entire TEX listing of mathematical and other symbols.

I.2 : Partitions of unity

(Conlon, §§ 1.4–1.5)

Additional exercises

1. (*i*) For each positive integer n let V_n be the open annulus (ring-shaped region) consisting of all points x such that $n - 2 < |x| < n + 1$. Prove that the family of subsets $\{ V_n \}$ is locally finite.

SOLUTION.

Given $x \in \mathbf{R}^n$, take the first positive integer k such that $|x| < k$. Then $N_k(0)$ is an open neighborhood of x and $N_k(0) \cap V_j = \emptyset$ if $j \geq k + 3$.■

(*ii*) Suppose X is a topological n-manifold and $\mathcal{U}$ is an open covering of X. Prove that there is an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that every subset of $\mathcal{V}$ is homeomorphic to $\mathbf{R}^n$.

SOLUTION.

Let $x \in X$ and pick U_α in $\mathcal{U}$ such that $x \in U_\alpha$. As indicated in the class notes, there is a neighborhood base of open neighborhoods for x such that each is homeomorphic to $\mathbf{R}^n$. Let V_x be such an open neighborhood for which $V_x \subset U_x$, and let $\mathcal{V}$ be the collectionof all such open sets V_x. Then $\mathcal{V}$ refines $\mathcal{U}$ and every set in $\mathcal{V}$ is homeomorphic to $\mathbf{R}^n$.

2. Suppose that U is open in $\mathbf{R}^n$ and W is an open neighborhood of $U \times \{0\}$ in $U \times [0, 1)$. Prove that there is an open subneighborhood $W_0 \subset W$ of U such that U is a deformation retract of W_0. [*Hint:* Partitions of unity guarantee the existence of a continuous positive valued function f on U whose graph is completely contained in W. Consider the open set W_0 of all points in $U \times [0, 1)$ that lie under the graph of f.]

SOLUTION.

Use the hint and take the function f described there. Let $j : U \to W$ be the composite of the canonical homeomorphism $U \cong U \times \{0\}$ with the inclusion of the latter in W, and let $q : W \to U$ be the restriction of the projection map from $U \times [0, 1)$ to U. Then $q \circ j = \mathrm{id}_U$, and $j \circ q$is homotopic to the identity by the homotopy $H(x, s; t) = (x, \ (1 - t)\, s)$; the crucial point is that the homotopy maps W to itself because $(x, t) \in W$ and $u \leq t$ imply $(x, u) \in W$.■

3. Let M be a second countable topological manifold and let $M^\bullet$ denote its one point compactification. Using the metrization theorems for topological manifolds, prove that $M^\bullet$ is also second countable and metrizable. [*Hint:* First prove that $A^\bullet$ is second countable because it is σ-compact. For metrizability, suppose more generally that A is a bounded locally compact subset of the normed vector space $\mathbf{R}^\infty$ and $\varphi : A \to [0, +\infty) \cong [0, 1)$ is a proper map. Consider the function $f : A^\bullet \to \mathbf{R}^\infty \times [0, 1]$ defined by

$$f(z, t) \quad = \quad \big([1 - \varphi(z)] \cdot f(z), \ 1 - \varphi(z) \big)$$

for ordinary points $z \in A$ and $f(\infty_A) = (0, 1)$. Prove that this map is continuous on all of A; there are two cases depending upon whether one has an ordinary point or ∞_A. Also verify that f is 1–1. Why do these properties suffice to show that f maps $A^\bullet$ homeomorphicall onto its image?]

SOLUTION.

The first thing to prove is that $A^\bullet$ is second countable. We know that A is second countable and open in $A^\bullet$ with $A^\bullet - A = \{ \infty_A \}$. Let $\mathcal{B}$ be a countable base for A. Since A is σ-compact, it follows that A is an increasing union of compact subsets K_n, and therefore the complements

$A^\bullet - K_n$ will form a countable neighborhood base $\mathcal{B}_\infty$ for ∞_A in $A^\bullet$. We claim that $\mathcal{B} \cup \mathcal{B}_\infty$ forms a base for the topology on $A^\bullet$. Since both $\mathcal{B}$ and $\mathcal{B}_\infty$ are countable, this base will be countable. To prove that $\mathcal{B} \cup \mathcal{B}_\infty$ iS a base, suppose that W is open in $A^\bullet$. There are two cases depending upon whether or not $\infty_A \in A$. If not, then W is a union of open sets in $\mathcal{B}$. If W contains ∞_A, then $A - W = C$ is compact and hence is contained in K_n for some n, which means that

$$\infty_A \quad \in \quad V_n \quad = \quad A^\bullet \; - \; K_n \quad \subset W \ .$$

Now $W \cap A$ is a union of sets in $\mathcal{B}$, and therefore it follows that $W = V_n \cup (A \cap W)$ is a union of open sets in $\mathcal{B} \cup \mathcal{B}_\infty$. This finishes the proof of second countability.

To prove the metrizability of $A^\bullet$, follow the hint. If A is metrizable, then as noted in the file `smirnov.*` we know that A is homeomorphic to a subset of a normed vector space E. In fact, if we compose the inclusion with the shrinking map σ from E to itself defined by

$$\sigma(v) \quad = \quad \frac{1}{1 + |v|} \cdot v$$

we obtain a homeomorphism f from A to a bounded subset of E; in fact, the image lies in $N_1(0)$. Following the hint further, we need to show that there is a proper map $\varphi : A \to [0, +\infty) \cong [0, 1)$; we know there is a proper map if $[0, 1)$ is replaced by $[0, +\infty)$ and we obtain a map into $[0, 1)$ by composing the known map to $[0, +\infty)$ with a homeomorphism from the latter to $[0, 1)$.

Once again following the hint, we now consider the function $f : A^\bullet \to E \times [0, 1]$ defined by

$$g(z) \quad = \quad \big([1 - \varphi(z)] \cdot f(z), \, 1 - \varphi(z) \big)$$

for ordinary points $z \in A$ and $f(\infty_A) = (0, 1)$. We claim this function is continuous; as suggested in the hint, we split this into two cases depending upon whether z is an ordinary point or ∞_A. For ordinary points this is an immediate consequence of the formula. At the point ∞_A, let K_n be the increasing sequence of compact sets as before, and take an arbitrary small neighborhood $W = N_\delta(0) \times (1 - \varepsilon]$ of $(0, 1) \in E \times [0, 1]$. Since φ is proper, we can find an $M > 0$ such that $x \notin K_M$ implies $\varphi(x) > 1 - \delta$ and $\varphi(x) > 1 - \varepsilon$. But this means that $x \in A - K_M$ implies that $g(x) \in W$, and hence we see that g is continuous at ∞_A.

We also claim that g is 1–1. For points in A this follows directly from the formula, and $g(\infty) \notin g(A)$ because the second coordinate of $g(\infty_A)$ is equal to 1 and points in $g(A)$ all have second coordinates strictly less than 1. Thus we have shown that g is a 1–1 continuous function from the compact Hausdorff space $A^\bullet$ into a metric space, and every such function is a homeomorphism onto its image.■

4. [*This question requires some background knowledge from measure theory.*] Given a second countable topological manifold X, define the family of *Borel sets* $\mathcal{B}$ in X to be the smallest family of subsets that contains the open subsets and is closed under the operations of countable union, countable intersection and complementation (so it follows that $\mathcal{B}$ also contains all closed subsets). We shall say that a nonnegative measure on $\mathcal{B}$ is *topologically well-behaved* if (i) all one point subsets have measure zero, (ii) all open subsets have positive measure. — For each $n > 0$, the standard Lebesgue measure on $\mathbf{R}^n$ defines a topologically well-behaved (Borel) measure.

(i) DISREGARD THIS PORTION OF THE PROBLEM.

(ii) If X is a topological n-manifold for some $n > 0$, prove that X has a topologically well-behaved Borel measure. [*Hint:* If U is an open subset of X and there is a homeomorphism $h : V \to U$ where

V is open in $\mathbf{R}^n$, why does h send the Borel subsets of V to the Borel subsets of U and vice versa? Show that one can define a measure m_U on U by setting $m_U(A) = |h^{-1}(A)|$ where $| \cdots |$ denotes the usual Lebesgue measure on $\mathbf{R}^n$. Why is this a topologically well-behaved measure? Finally, show that if one pieces a suitable collection of such local measures together using a partition of unity then one obtains a Borel measure with the desired properties.]

SOLUTION.

We shall follow the hint except that the last part is a bit misleading. One wants to piece local measures together in order to get a global measure, but partitions of unity are not particularly helpful. *We shall also assume that U is a* **bounded** *open subset of* $\mathbf{R}^n$.

To see that h sends Borel sets in V to Borel sets in U, we first note that the homeomorphism h has the important algebraic properties for subsets $E_\alpha \subset V$:

$$\begin{aligned} h(\cup_\alpha\, E_\alpha) &= \cup_\alpha\, h(E_\alpha) \\ h(\cap_\alpha\, E_\alpha) &= \cap_\alpha\, h(E_\alpha) \\ h(V\ -\ E_\alpha) &= U\ -\ h(E_\alpha) \end{aligned}$$

These equations have the following basic consequence: *If $\mathcal{M}_V$ is the family of Borel sets in V then the family $h_*\mathcal{M}_V$ of all $D \subset U$ such that $D = h(E)$ for some Borel subset E, then $h_*\mathcal{M}_V$ is closed under countable unions, countable intersections and complements.* Since h is a homeomorphism, this family contains all open subsets of U, and since the family $\mathcal{M}_U$ of Borel subsets is the smallest family of subsets that contains all open sets and is closed under countable unions, countable intersections and complementation, it follows that $\mathcal{M}_U$ is contained in $h_*\mathcal{M}_V$. Conversely, since $h_*\mathcal{M}_V$ is the smallest family that contains all sets of the form $h(W)$, where W open in V, and it is closed under countable unions, countable intersections and complementation, it also follows that $h_*\mathcal{M}_V$ is contained in $\mathcal{M}_U$. Therefore the two families of sets are equal, and this means that Borel sets in V are mapped to Borel sets in U and vice versa.

If V is open in $\mathbf{R}^n$, define a function m_U on $\mathcal{M}_U$ by setting $m_U(A) = |h^{-1}(A)|$, where $| \cdots |$ denotes the usual Lebesgue measure on $\mathbf{R}^n$; this can be done because $h_*\mathcal{M}_V = \mathcal{M}_U$ implies that $h^{-1}(A) \in \mathcal{M}_V$ if $A \in \mathcal{M}_U$. This takes values in the nonnegative reals (boundedness implies that all subset of U have finite measure!) and it is countably additive: If $E = \cup_n E_n$ where the sets E_n are pairwise disjoint Borel sets in V, then their images are pairwise disjoint Borel sets in U and hence

$$m_U(\cup_n\, E_n) \ = \ |\, h^{-1}(\cup_n\, E_n)\,| \ = \ |\, \cup_n \ h^{-1}(E_n)\,| \ = \ \sum_n \ |\, h^{-1}(E_n)\,| \ = \ \sum_n \ m_U(E_n)$$

so that m_U is countably additive. The condition $m_U(\emptyset) = 0$ follows also because $h^{-1}(\emptyset) = \emptyset$.

This is a topologically well-behaved measure because $m_U(\{x\}) = |\,\{h^{-1}(x)\}\,| = 0$ and if W is a nonempty open set in W then $m_U(W) = |\,h^{-1}(W)\,| > 0$ because $h^{-1}(W)$ is a nonempty open set in V and the Lebesgue measure of a nonempty open set is always positive.

Finally, suppose that X is a topological n-manifold for some $n > 0$, and let $\mathcal{U}$ be a countable open covering of X by subsets that are homeomorphic to open subsets of $\mathbf{R}^n$. In fact, every point has an open neighborhood that is homeomorphic to a bounded subset of $\mathbf{R}^n$ with Lebesgue measure ≤ 1. For each U_j in $\mathcal{U}$ let $h_j : V_j \to U_j$ be a homeomorphism from an open subset V_j in $\mathbf{R}^n$ that is bounded with Lebesgue measure ≤ 1, and let μ_j be the measure on U_j constructed and studied in the preceding paragraphs. Given a Borel set $E \subset X$, define

$$\mu(E) \quad = \quad \sum_j \; 2^{-j} \cdot \mu_j(E \cap U_j)$$

(this definition is valid because if E is a Borel set then $E \cap U_j$ is a Borel subset of U_j; note also that the infinite series converges).

We need to show that μ is countably additive, its value on a one point set is always zero, and its value on a nonempty open subset is always positive. To verify countable additivity, note that if the family of Borel sets E_k is pairwise disjoint then for each j the same is true for $E_k \cap U_j$ and therefore we have

$$\mu(\cup_k \; E_k) \quad = \quad \sum_j \; 2^{-j} \cdot \mu_j(\cup_k \; E_k \cap U_j) \quad =$$

$$\sum_{j,k} \; 2^{-j} \cdot \mu_j(E_k \cap U_j) \quad = \quad \sum_k \; \mu(E_k) \; .$$

The fact that $\mu(\{x\}) = 0$ follows because $\{x\} \cap U_j$ is either a single point or the empty set, so that in all cases $\mu_j(\{x\} \cap U_j) = 0$. Finally, suppose that W is a nonempty open subset of X. Then there is some J such that $W \cap U_J$ is nonempty (the sets U_j form an open covering), and therefore we have

$$0 \quad < \quad 2^{-J} \cdot \mu_J(W \cap U_J) \quad \leq \sum_j \; 2^{-j} \cdot \mu_j(W \cap U_j) \quad = \quad \mu(W) \; .$$

Therefore μ is a topologically well-behaved measure on the Borel subsets of X.∎

I.3 : The Contraction Lemma

Additional exercises

1. Prove that the equation $2 - x - \sin x = 0$ has a real root and that it lies in the closed interval with endpoints $\pi/6$ and $\pi/2$. Show that $\varphi(x) = 2 - \sin x$ is a contraction operator on this interval and then find the root, accurate to six decimal places. [In this example it might be worthwhile to use a scientific calculator to estimate the numerical value.]

SOLUTION.

We need to show that φ maps $[\pi/6, \pi/2]$ into itself and that the absolute value of its derivative takes a maximum value that is less than 1.

By construction φ is decreasing, and its values range from 1 to 3/2. Since $3 < \pi < 4$ we know that

$$\frac{\pi}{6} \quad < \quad \frac{2}{3} \quad < 1 \quad < \quad \frac{3}{2} \quad < \quad \frac{\pi}{2}$$

and therefore φ does map the interval in question into itself.

The derivative of φ is $-\cos x$, and on the interval in question its values lie in the interval $[0, \sqrt{3}/2]$, so the maximum of the absolute value of φ' is less than 0.9, which suffices for our purposes.

If one uses a scientific calculator to estimate the solution of this equation starting with the trial value $x = \pi/2$ one obtains an approximation to 6 decimal places after 18 iterations. The convergence is slower here because the upper bound on $|\varphi'|$ is higher. Of course, when doing this it is necessary to remember that angle measurements must use the radians setting on the calculator.

Here is an explicit table of the successive approximations:

$\pi/2$
1.000000
1.158529
1.108379

1.116264
1.101533
1.108098
1.105149

1.106469
1.105877
1.106142
1.106023

1.106077
1.106053
1.106065
1.106059

1.106060
1.106060
etc.

The final line indicates that one will get the same answer for all subsequent iterations.■

2. Let X be a metric space. A map $f : X \to X$ is said to be a *nonisometric* (or proper) *similarity* of X if f is onto and there is a positive constant $C \neq 1$ such that

$$\mathbf{d}\,(\,f(u),\,f(v)\,) \quad = \quad C \cdot \mathbf{d}(u,v)$$

for all $u,\,v \in X$ (hence f is 1–1 and uniformly continuous, and in fact has a uniformly continuous inverse that is also a proper similarity). Prove that every nonisometric similarity of a complete metric space has a unique fixed point. [*Hint and comment:* Split into two cases depending upon whether $C < 1$ and $C > 1$. In the first case the surjectivity condition turns out to be unnecessary. In the second case, verify that f has an inverse that is uniformly continuous. Why does $f(x) = x$ hold if and only if $f^{-1}(x) = x$? — The most elementary examples of such maps arise when $X = \mathbf{R}^n$ and a classical geometric similarity is given by $f(x) = cAx + b$, where A comes from an orthogonal matrix and either $0 < C < 1$ or $C > 1$; for these examples one can prove the existence of a unique fixed point using elementary linear algebra.]

SOLUTION.

First of all, the map f is 1–1 onto; we are given that it is onto, and it is 1–1 because $u \neq v$ implies $\mathbf{d}(\,f(u),\,f(v)\,) > \mathbf{d}(u,v) > 0$. Therefore f has an inverse, at least set-theoretically, and we denote f^{-1} by T.

We claim that T satisfies the hypotheses of the Contraction Lemma. The proof of this begins with the relations

$$\mathbf{d}(\,T(u),\,T(v)\,) = \mathbf{d}(\,f^{-1}(u),\,f^{-1}(v)\,) = \frac{1}{C1}\,\mathbf{d}\left(\,f\left(f^{-1}(u)\right),\,f\left(f^{-1}(v)\right)\,\right) = \mathbf{d}(u,v)\ .$$

Since $C > 1$ it follows that $0 < 1/C < 1$ and consequently the hypotheses of the Contraction Lemma apply to our example.

Therefore T has a unique fixed point p; we claim it is also a fixed point for f. We shall follow the hint. Since T is 1–1 and onto, it follows that $x = T(T^{-1}(x))$ and that $T(x) = x \implies x = T^{-1}(x)$; the converse is even easier to establish, for if $x = T^{-1}(x)$ the application of T yields $T(x) = x$. Since there is a unique fixed point p such that $T(p) = p$, it follows that there is a unique point, in fact the same one as before, such that $p = T^{-1}(p)$, which is equal to $f(p)$ by definition.■

THE CLASSICAL EUCLIDEAN CASE.

This has two parts. The first is that every expanding similarity of $\mathbf{R}^n$ is expressible as a so-called affine transformation $T(v) = cAv + b$ where A is given by an orthogonal matrix. The second part is to verify that each transformation of the type described has a unique fixed point. By the formula, the equation $T(x) = x$ is equivalent to the equation $x = cAx + b$, which in turn is equivalent to $(I - cA)x = b$. The assertion that T has a unique fixed point is equivalent to the assertion that this linear equation has a unique solution. The latter will happen if $I - cA$ is invertible, or equivalently if $\det(I - cA) \neq 0$, and this is equivalent to saying that c^{-1} is not an eigenvalue of A. But if A is orthogonal this means that $|Av| = |v|$ for all v and hence the only possible eigenvalues are ± 1; on the other hand, by construction we have $0 < c^{-1} < 1$ and therefore all of the desired conclusions follow. The same argument works if $0 < c < 1$, the only change being that one must substitute $c^{-1} > 1$ for $0 < c^{-1} < 1$ in the preceding sentence.■

I.4 : Basic topological constructions revisited

(*cf.* [MUNKRES1], §§ 15, 16, 19, 22)

Additional exercises

1. (*i*) Let X, Y and Z be sets (*resp.*, topological spaces), and let $\times$ denote the usual cartesian product. Prove that

$$X \times (Y \times Z)$$

is a direct product of sets (*resp.*, topological spaces) as defined in the notes.

SOLUTION.

See the comment after (*ii*).

(*ii*) Let A, B, C and D be sets (*resp.* topological spaces), and let $\times$ denote the usual cartesian product. Prove that

$$(A \times B)\ \times\ (C \times D)$$

is a direct product of sets (*resp.* topological spaces) as defined in the notes.

SOLUTION.

These are special cases of Exercise 5 below (*cf.* the remark below), so we simply refer to the solution of the latter.■

[*Note:* These may all be viewed as special cases of a more general result.]

2. (i) Let X and Y be topological spaces and let $\tau : X \times Y \to Y \times X$ be the "twist map" which sends (x,y) to (y,x) for all x and y. Prove that τ is a homeomorphism. [*Hint:* Consider the analogous map $\tau' : Y \times X \to X \times Y$.]

SOLUTION.

If $\pi_2^{X \times Y} : X \times Y \to Y$ is projection onto the second factor and $\pi_1^{X \times Y} : X \times Y \to X$ is projection onto the first factor, then by the Universal Mapping Property there is a unique continuous map $\tau_{X,Y} : X \times Y \to Y \times Y$ such that $\pi_1^{Y \times X} \circ \tau_{X,Y} = \pi_2^{X \times Y}$ and $\pi_2^{Y \times X} \circ \tau_{X,Y} = \pi_1^{X \times Y}$. These formulas immediately imply that $\tau_{X,Y}(u,v) = (v,u)$. We claim that $\tau_{X,Y}$ is a homeomorphism, and in fact its inverse is equal to $\tau_{Y,X}$. To see this, first note that $\tau_{Y,X}(v,u) = (u,v)$, and then use this to check that. $\tau_{Y,X} \circ \tau_{X,Y}(u,v) = (u,v)$ and $\tau_{X,Y} \circ \tau_{Y,X}(v,u) = (v,u)$ for all u and v.■

(ii) Let X be a topological space and let $T : X \times X \times X \to X \times X \times X$ be the map that cyclically permutes the coordinates: $T(x,y,z) = (z,x,y)$. Prove that T is a homeomorphism. [*Hint:* What is the test for continuity of a map into a product? Can you write down an explicit formula for the inverse function?]

SOLUTION.

Let $\pi_j : X \times X \times X \to X$ be projection onto the j^{rmth} factor for $j = 1, 2, 3$. The map T is the unique one such that $\pi_1 \circ T = \pi_3$, $\pi_2 \circ T = \pi_1$ and $\pi_3 \circ T = \pi_2$. Since the projections of T onto the coordinates are continuous, it follows that T is continuous. To see that T is a homeomorphism, observe that

$$T \circ T \circ T(x,y,z) \quad = \quad (x,y,z)$$

for all (x,y,z) by direct calculation. This implies that the continuous map $T \circ T$ is an inverse to T, and therefore T is a homeomorphism.■

3. ("A product of products is a product.") Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of nonempty sets, and let $\mathcal{A} = \cup\{\mathcal{A}_\beta \mid \beta \in \mathcal{B}\}$ be a partition of $\mathcal{A}$. Construct a bijective map of $\prod\{A_\alpha \mid \alpha \in \mathcal{A}\}$ to the set

$$\prod_\beta \{\prod\{A_\alpha \mid \alpha \in \mathcal{A}_\beta\} \} \ .$$

If each A_α is a topological space and we are working with product topologies, prove that this bijection is a homeomorphism.

SOLUTION.

We shall use the characterization of products by the Universal Mapping Property for Cartesian products and work simultaneously with sets and topological spaces, and morphisms between such objects will mean set-theoretic functions or continuous functions in the respective cases.

For each β let $\prod_\beta$ denote the product of objects whose index belongs to $\mathcal{A}_\beta$ and denote its coordinate projections by p_α. The conclusions amount to saying that there is a canonical morphism from $\prod_\beta P_\beta$ to $\prod_\alpha A_\alpha$ that has an inverse morphism. Suppose that we are given morphisms f_α from the same set S to the various sets A_α. If we gather together all the morphisms for indices α lying in a fixed subset $\mathcal{A}_\beta$, then we obtain a unique map $g_\beta : S \to P_\beta$ such that $p_\alpha \circ g_\beta = f_\alpha$ for all $\alpha in \mathcal{A}_\beta$. Let $q_\beta : \prod_\gamma P_\gamma \to P_\beta$ be the coordinate projection. Taking the maps g_β that have been constructed, one obtains a unique map $F : S \to \prod_\beta P_\beta$ such that $q_\beta \circ F = g_\beta$ for all β. By construction we have that $p_\alpha \circ q_\beta \circ F = f_\alpha$ for all α. If there is a unique map with this property, then $\prod_\beta P_\beta$ will be isomorphic to $\prod_\alpha A_\alpha$ by the lemma. But suppose that θ is any map with this property. Once again fix β. Then $p_\alpha \circ q_\beta \circ F = p_\alpha \circ q_\beta \circ \theta = f_\alpha$ for all $\alpha \in \mathcal{A}_\beta$ implies that $q_\beta \circ F = q_\beta \circ \theta$, and since the latter holds for all β it follows that $F = \theta$ as required.■

4. Let A be some nonempty set, let $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ and $\{Y_\alpha \mid \alpha \in \mathcal{A}\}$ be families of topological spaces, and for each $\alpha \in A$ suppose that $f_\alpha : X_\alpha \to Y_\alpha$ is a homeomorphism. Prove that the product map

$$\prod_\alpha f_\alpha \;:\; \prod_\alpha X_\alpha \longrightarrow \prod_\alpha Y_\alpha$$

is also a homeomorphism. [*Hint:* What happens when you take the product of the inverse maps?]

SOLUTION.

For each α let $g_\alpha = f_\alpha^{-1}$. Then we have

$$\prod_\alpha f_\alpha \circ \prod_\alpha g_\alpha \;=\; \prod_\alpha (f_\alpha \circ g_\alpha) \;=\; \prod_\alpha \mathrm{id}(Y_\alpha) \;=\; \mathrm{id}\left(\prod_\alpha Y_\alpha\right)$$

and we also have

$$\prod_\alpha g_\alpha \circ \prod_\alpha f_\alpha \;=\; \prod_\alpha (g_\alpha \circ f_\alpha) \;=\; \prod_\alpha \mathrm{id}(X_\alpha) \;=\; \mathrm{id}\left(\prod_\alpha X_\alpha\right)$$

so that the product of the inverses $\prod_\alpha g_\alpha$ is an inverse to $\prod_\alpha f_\alpha$.■

5. (i) Suppose that $f : X \to Y$ and $g : Y \to Z$ are topological embeddings. Prove that $g \circ f$ is also a topological embedding.

SOLUTION.

We shall show that $g \circ f$ has the Universal Mapping Property. Suppose that we are given a continuous map $h : W \to Z$ such that $h(W) \subset g \circ f(X)$. Then there is a unique map $h' : W \to Y$ such that $g \circ h' = h$. We claim that $h'(W) \subset f(X)$; this is true because $w \in W \Longrightarrow g(h'(w)) = h(w) = g(f(x))$ for some x, and since g is 1–1 these equations imply $h'(w) = f(x)$. Therefore since f is an embedding there is a unique map $h'' : W \to X$ such that $f \circ h'' = h'$. By construction the map h'' also satisfies $g \circ f \circ h'' = h$, and we claim it is the only such continuous map. But if k were another such map, then we would have $g \circ f \circ k = h = g \circ f \circ h''$, and since both g and f are 1–1 this would imply $k = h''$. Therefore $g \circ f$ has the desired Universal Mapping Property.■

(ii) Suppose that $h : A \to X$ and $k : B \to Y$ are topological embeddings. Prove that $h \times k : A \times B \to X \times Y$ is also a topological embedding.

SOLUTION.

Since h and k are topological embeddings this means that h maps A homeomorphically onto $h(A)$ and k maps B homeomorphically onto $k(B)$. It follows that $h \times k$ maps $A \times B$ homeomorphically onto $h(A) \times k(B) = h \times k(A \times B)$.■