Math 205C
Spring 2003

Take home assignment 1

Due Wednesday, May 4, 2005

1. Let U be an open subset of $\mathbf{R}^n$, let $\mathcal{A}$ be a smooth atlas for U containing the standard chart (U, J), were J is the identity map on U, and let (U, h) be an arbitrary smooth chart in $\mathcal{A}$.

(i) How can one express the transition map "$J^{-1}h$" in terms of h?

SOLUTION.

The transistion map is simply h because the value of "$J^{-1}h$" on a typical point x is the unique y such that $J(y) = h(x)$. Since J is the identity map, this means that $y = h(x)$.■

(ii) Why does this imply that h is smooth?

SOLUTION.

The transition maps in a smooth atlas are smooth, and since h equals the transition map "$J^{-1}h$" it follows that h must be smooth.■

2. Suppose that $f : M \to M'$ ahd $g : N \to N'$ are smooth maps. Prove that the map $f \times g : M \times N \to M' \times N'$ defined by $f \times g(x, y) = (f(x), g(y))$ is smooth. [*Hint* Let p_1 and p_2 be the projections from $M \times N$ to M and N respectively, and similarly let q_1 and q_2 be the projections from $M' \times N'$ to M' and N' respectively. Consider the composites of $f \times g$ with q_1 and q_2.]

SOLUTION.

Let π_1 and π_2 denote the projections of $N \times N'$ onto N and N' respectively, and let ϕ_1 and ϕ_2 denote the projections of $M \times M'$ onto M and M' respectively. Then by construction we have $\pi_1 \circ (f \times g) = f \circ \varphi_1$ and $\pi_2 \circ (f \times g) = g \circ \varphi_2$ so that the coordinate projections of $f \times g$ are smooth and hence $f \times g$ itself is smooth.■

3. Suppose that M is a topological 2-manifold. Prove that for each point $x \in M$ there is a neighborhood base $\{U_\alpha\}$ such that for each α we have

(i) The set $U_\alpha - \{x\}$ is connected, and the fundamental group of $U_\alpha - \{x\}$ with respect to some (in fact any) basepoint is nontrivial.

(ii) If $U_\beta \subset U_\alpha$ and $y \in U_\beta$ is different from x, then the includion of $U_\beta - \{x\}$ in $U_\alpha - \{x\}$ gives rise to an isomorphism of fundamental groups.

SOLUTION.

It suffices to show this for an open disk $N_r(v) \subset \mathbf{R}^2$, for if it is true in this case and one has an open neighborhood of a point x in a 2-manifold that is homeomorpic to $N_r(v)$ by some homeomorphism h, then one can prove the theorem for x and M by taking the images of the neighborhoods of v under the map h.

Let U_n be the disk of radius $1/n$ centered at v for all n such that $n > 1/r$. Then $\pi_1(U_n - \{v\})$ is infinite cyclic. and the neighborhoods U_n form a neighborhood base at v.

It remains to prove the second property. Supppose that $m > n$, and let C_m denote the circle of radius $1/m$ centered at v. Then C_m is a strong deformation retract of both $U_n - \{v\}$ and $U_m - \{v\}$

(this is just material out of 205B), and thus we know that the inclusions $a : C_m \to U_m - \{v\}$ and $b : C_m \to U_n - \{v\}$ determine isomorphisms of fundamental groups. If $c : U_m - \{v\} \to U_n - \{v\}$ is the remaining inclusion map, then $b = c \circ a$, which implies that $b_* = c_* \circ a_*$ on fundamental groups, which in turn implies that c_* is the isomorphism $b_* \circ (a_*)^{-1}$.■

4. Suppose that M is a topological n-manifold for some $n \geq 3$.

(i) Prove that for each point $x \in M$ there is a neighborhood base $\{U_\alpha\}$ such that U_α is simply connected.

SOLUTION.

As before we reduce to the case of an open disk $N_r(v)$ about a point v. Let U_n be as before, and let S_m be the sphere of points whose distance from v is equal to $1/m$. As in the previous exercise S_m is a strong deformation retract of $U_n - \{v\}$ and $U_m - \{v\}$. however, in this case S_m is simply connected and therefore the sets $U_n - \{v\}$ and $U_m - \{v\}$ are also simply connected,■

(ii) Explain why the conditions

M is a topological 1-manifold,

M is a topological 2-manifold,

M is a topological 3-manifold,

are mutually exclusive without using Brouwer's Invariance of Domain or Dimension theorems.

SOLUTION.

In a 1-manifold, for every sufficiently small neighborhood U of a point v the set $U - \{v\}$ is disconnected, while in a manifold of higher dimension there is always a neighborhood base U_m such that the sets $U_m - \{v\}$ are always connected. Therefore a space cannot be a 1 manifold and simlutaneously a k-manifold for any $k \geq 2$.

We only need to show that a space cannot be both a 2-manifold and a 3-manifold. The idea is to take the preceding part of this exercise along with the preceding exercise to obtain a contradiction. Assume that a space is a 2-manifold, and let U_α be the neighborhood base given by Exercise 3. Now assume that it is also a 3-manifold. Given α we can then find an open subset $W \subset U_\alpha$ such that $W - \{v\}$ is simply connected. Choose $U_\beta \subset U_\alpha$ such that the inclusion of U_β in U_α defines an isomorphism of nontrivial fundamental groups. Since the inclusion maps factor as composites

$$U_\beta - \{v\} \quad \subset \quad W - \{v\} \quad \subset \quad U_\alpha - \{v\}$$

there is a corresponding factorization of the fundamental group homomorphisms. Since the middle space is simply connected, it follows that this homomorphism must be the trivial homomorphism. However, by construction this homomorphism is an isomorphism from one nontrivial group to another, and thus we have a contradiction. The problem arose from our assumption that the space was a topological 3-manifold, which implied the existence of the simply connected deleted neighborhood $W - \{v\}$. Therefore the 2-manifold we started with cannot also be a 3-manifold.■

Sketch of shortcut. The previous exercise has the following consequence3 for 2-manifolds. *For each $x \in M$ there is an open neighborhood $V-)$ such that for all connected neightborhoods V of x such that $V \subset V_0$ the deleted neighborhood $V_0 - \{x\}$ is not simply connected.* — This is inconsistent with the existence of a neighborhood base at x such that the corresponding deleted neighborhoods are all simply connected.■