SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 3

Fall 2003

II. Local theory of smooth functions

II.1 : Differentiability

1. Use the derivative approximation to estimate the following:

(*i*) $[(3.02)^2 + (1.97)^2 + (5.98)^2]$

(*ii*) $(e^4)^{1/10} = \exp\left((1.1)^2 - (0.9)^2\right)$

SOLUTION.

Take $f(x,y,z) = x^2 + y^2 + z^2$, so that $\nabla f(v) = 2v$. The general approximation rule is

$$h(v + \Delta v) \approx h(v) + \langle \nabla h(v), \Delta v \rangle$$

and in this special case $h = f$, $v = (3,2,6)$ and $\Delta v = (0,02,-0.03,-0.02)$. Thus the formula specializes to

$$[(3.02)^2 + (1.97)^2 + (5.98)^2] \approx [3^2 + 2^2 + 6^2] + \langle (6,4,12)\,,\,(0,02,-0.03,-0.02)\rangle$$

which simplifies to 48.76; for the sake of comparison, we note that the actual value is 48.7617 .■

(b) $(e^4)^{1/10} = \exp\left((1.1)^2 - (0.9)^2\right)$

SOLUTION.

Take $f(x,y) = x^2 - y^2$, so that $\nabla f(x,y) = (2xe^{x^2-y^2},\ -2ye^{x^2-y^2})$. This provides some of the substitutions needed in the general approximation rule described above. The remaining pieces are $v = (1,1)$, $f!,1) = 1$ and $\Delta v = (0.1, -0.1)$. Thus the formula specializes to

$$(e^4)^{1/10} = \exp\left((1.1)^2 - (0.9)^2\right) \approx e^{1^2-1^2} + 4\cdot(0.1)\cdot e^{1^2-1^2} = 1.4$$

For the sake of comparison, we note that the actual value is 1.4918247 to seven decimal places.■

2. Let $f : \mathbf{R}^n \to \mathbf{R}$ be differentiable. If $f(0) = 0$ and $f(tx) = tf(x)$ for all t and x prove that $f(x) = \langle \nabla f(0), x\rangle$ for all x; *i.e.*, f is linear. Consequently, any nonlinear function g satisfying the conditions $g(0) = 0$ and $g(tx) = t\,g(x)$ for all t and x is not differentiable although it has directional derivatives in all directions at the origin (why?).

SOLUTION.

Let $x \in \mathbf{R}^n$ be an arbitrary vector; then

$$[Df(0)](x) \quad = \quad \lim_{t\to 0} \frac{1}{t}\cdot\left(f(tx) - f(0)\right) \quad = \quad \lim_{t\to 0} f(x)$$

and the latter is just $f(x)$. Since the left hand side is a linear function of u the same is true of the right hand side. But this means that $f(x) = \langle \nabla f(0), x \rangle$ for all x.

If g is a nonlinear function satisfying the conditions $g(0) = 0$ and $g(tx) = tg(x)$ for all t and x, the right hand side of the displayed formula shows that g has directional derivatives in all directions through the origin, but if it were differentiable it would have to be linear. Therefore g cannot be linear.■

3. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x,y) \quad = \quad \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

(*i*) Show that $D_1 f(0,y) = -y$ and $D_2 f(x,0) = x$ for all x and y.

(*ii*) Conclude that $D_1 D_2 f(0,0)$ and $D_2 D_1 f(0,0)$ exist but are not equal.

SOLUTION.

We have

$$D_1 f(0,y) \quad = \quad \lim_{t \to 0} \frac{1}{t} \, \frac{ty(t^2 - y^2)}{t^2 + y^2} \quad = \quad \lim_{t \to 0} \frac{yt^2 - y^3}{t^2 + y^2}$$

for all y. If $y \neq 0$ then this limit can easily be evaluated as $-y$. On the other hand if $y = 0$ then we have $f(x,0) = 0$ so that the first partial is $0 = -y$.■

(*ii*) Conclude that $D_1 D_2 f(0,0)$ and $D_2 D_1 f(0,0)$ exist but are not equal.

SOLUTION.

Since $D_2 f(x,0) = x$ it follows that $D_1 D_2 f(0,0) = 1$, and since $D_1 f(0,y) = -y$ it follows that $D_2 D_1 f(0,0) = -1$.■

FOOTNOTE.

One easy way to see the continuity of f at the origin is to write it in terms of polar coordinate; in these terms the value of the function is

$$\frac{r^2 \sin 4\theta}{4}$$

(verify this), and continuity at the origin is clear from this reformulation.■

4. Show that each of the following is a solution of the *heat equation*

$$\frac{\partial u}{\partial t} \quad = \quad k^2 \frac{\partial^2 u}{\partial x^2}$$

(where k is a constant):

(*i*) $\exp(-k^2 a^2 t) \sin ax$

SOLUTION.

Direct computations show that both sides of the partial differential equation are equal to $-k^2 a^2 \exp(-k^2 a^2 t) \sin ax$.■

(*ii*) $\exp(-x^2/4k^2t)/\sqrt{t}$

SOLUTION.

Direct but significantly more tedious computations show that both sides of the partial differential equation are equal to

$$\frac{x^2 - 2k^2}{4k^2 t^{3/2}} \exp(-x^2/4k^2t) \ .$$

5. (*i*) If $f(x) = g(\rho)$ where $\rho = |x|$ and the number n of variables is at least 3, show that

$$\nabla^2 f \quad = \quad \frac{n-1}{\rho}\, g'(\rho) + g''(\rho)$$

for $x \neq 0$.

SOLUTION.

We need to compute the second partials with respect to each variable x_i and add them up. By the Chain Rule

$$\frac{\partial f}{\partial x_i} \quad = \quad g'(\rho)\, \frac{\partial \rho}{\partial x_i}$$

and the second factor of the right hand side is equal to

$$\frac{x_i}{\rho}$$

because $\rho = \left(\sum_j x_j^2\right)^{1/2}$. We must next differentiate this function

$$\frac{x_i\, g'(\rho)}{\rho}$$

once again with respect to x_i and see what happens.

$$\frac{\partial^2 f}{\partial x_i^2} \quad = \quad \frac{\partial}{\partial x_i}\left(\frac{x_i\, g'(\rho)}{\rho}\right) \quad = \quad x_i\, \frac{\partial}{\partial x_i}\left(\frac{g'(\rho)}{\rho}\right) \quad + \quad \frac{g'(\rho)}{\rho} \quad =$$

$$\frac{d}{d\rho}\left(\frac{g'(\rho)}{\rho}\right)\frac{x_i^2}{\rho} \quad + \quad \frac{g'(\rho)}{\rho} \quad = \quad \left(\frac{g''(\rho)}{\rho} - \frac{g'(\rho)}{\rho^2}\right)\frac{x_i^2}{\rho} \quad + \quad \frac{g'(\rho)}{\rho}$$

If we add these expressions over all i such that $1 \leq i \leq n$, we obtain the Laplacian of f. Since

$$\sum_{i-1}^{n} \frac{x_i^2}{\rho} = \rho$$

(recall that $\sum_i x_i^2 = \rho^2$) the sum of all the second partial derivatives that gives the Laplacian is equal to

$$\sum_{i-1}^{n} \left(\frac{g''(\rho)}{\rho} - \frac{g'(\rho)}{\rho^2}\right)\frac{x_i^2}{\rho} \quad + \quad \frac{g'(\rho)}{\rho}$$

which simplifies to

$$g''(\rho) - \frac{g'(\rho)}{\rho} + \frac{n\, g'(\rho)}{\rho}$$

and the latter simplifies further to the expression at the end of the exercise.■

(*ii*) Using the formula displayed above, prove that if $\nabla^2 f = 0$ then

$$f(x) \quad = \quad \frac{a}{|x|^{n-2}} + b$$

where $x \neq 0$ and a and b are constants.

SOLUTION.

By the formula from the preceding exercise we have

$$\frac{n-1}{\rho}\, g'(\rho) \ + \ g''(\rho) \quad = \quad 0$$

which reduces to a separable first order differential equation in g' whose associated general solution for $g(\rho) = f(x)$ has the indicated form.■

6. Verify that the functions $r^n \cos^n \theta$ and $r^n \sin^n \theta$ satisfy the 2-dimensional Laplace equation in polar coordinates. [Exercise 3.9 on the same page gives the formula for the Laplacian in polar coordinates.

SOLUTION.

The polar form of the Laplacian is

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{1}{r}\frac{\partial g}{\partial r} \ .$$

If we substitute $g(r,\theta) = r^n \cos^n \theta$ into this expression we obtain

$$n(n-1)\, r^{n-2}\cos n\theta \ - \ n^2 r^{n-2}\cos n\theta \ + \ n r^{n-2}\cos n\theta$$

which is equal to 0.■

7. If

$$f(x,y,z) \quad = \quad \frac{1}{\rho}\cdot g\left(t - \frac{\rho}{c}\right)$$

where $\rho = (x^2 + y^2 + x^2)^{1/2}$ and c is a constant, show that f satisfies the 3-dimensional *wave equation*

$$\nabla^2 f \quad = \quad \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} \ .$$

SOLUTION.

The idea again is to compute both sides explicitly and note that they are equal. For this purpose it will be helpful to use the formula for the Laplacian in spherical coordinates that is given in Exercise 3.10(b) on the same page as the problem we are working. If we write $f(x,y,z)$ as $F(\rho,\theta,\phi)$ and F does not depend upon θ or ϕ (as in our case), the Laplacian formula in spherical coordinates reduces to

$$\nabla^2 f \quad = \quad \frac{\partial^2 F}{\partial \rho^2} \ + \ \frac{2}{\rho}\frac{\partial F}{\partial \rho} \ .$$

Computing the right hand side of the wave equation for our choice of f is trivial:

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} \quad = \quad \frac{1}{c^2}\frac{g''(t-\rho/c)}{\rho}$$

What about the left hand side? Of course the first step is to compute the partial derivative with respect to ρ:

$$\frac{\partial F}{\partial \rho} \quad = \quad \frac{1}{\rho}\frac{(-1)}{c}\,g'(t-\rho/c) \;-\; \frac{1}{\rho^2}\,g(t-\rho/c)$$

If we take partial derivatives with respect to ρ again we obtain the following:

$$\frac{\partial^2 F}{\partial \rho^2} \quad = \quad \frac{1}{\rho}\frac{1}{c^2}\,g''(t-\rho/c) + \frac{1}{\rho^2}\frac{1}{c}\,g'(t-\rho/c) - \frac{1}{\rho}\frac{(-1)}{c}\,g'(t-\rho/c) + \frac{2}{\rho^3}\,g(t-\rho/c)$$

If we substitute these into the expression for the Laplacian in spherical coordinates we obtain the same function that we obtained for the right hand side of the wave equation.■

8. The following shows the hazards of denoting functions by real variables. Let $w = f(x,y,z)$ and $z = g(x,y)$. Then

$$\frac{\partial w}{\partial x} \quad = \quad \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} \quad = \quad \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}$$

because the partials of x and y with respect to x are 1 and 0 respectively. Therefore

$$\frac{\partial w}{\partial z}\frac{\partial z}{\partial x} = 0$$

But if $w = x + y + z$ and $z = x + y$ then the expression on the left hand side is $1 \cdot 1 = 1$, so that $0 = 1$. Where is the mistake?

SOLUTION.

Since the point of this fallacy is to show the need to write things down less casually, we should begin by doing so. The symbol w is actually being used for two separate functions; namely, $f(x,y,z)$ and $B(x,y) = f(x,y,\,g(x,y)\,)$. The application of the Chain Rule in the first line then becomes

$$\frac{\partial B}{\partial x} \quad = \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}$$

which is consistent with the previous displayed line. The mistake leading to the fallacy is that one cannot assume that the partial derivatives of B and f with respect to x are equal; these are two separate functions, and in fact the given example illustrates this fact very clearly.■

9. Let α and β be norms on $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively. Prove that $\gamma_0(x,y) = \alpha(x) + \beta(y)$ and $\gamma_1(x,y) = \max(\alpha(x),\,\beta(y)\,)$ define norms on $\mathbf{R}^{m+n} \cong \mathbf{R}^m \times \mathbf{R}^n$.

SOLUTION.

In the section on Cartesian products in the course notes we showed that functions of this sort defined metrics on a product. The proof that the functions described here are norms is similar.■

10. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a $1-1$ linear mapping. Prove that there is an $\varepsilon > 0$ such that if $S : \mathbf{R}^n \to \mathbf{R}^m$ is linear and satisfies $\|S - T\| < \varepsilon$, then S is also $1-1$.

SOLUTION.

By the results on equivalences of norms, it suffices to prove the result for any norm on the space of $m \times n$ matrices. For the sake of definiteness we shall use the norm associated to the standard inner product on the space of all $m \times n$ matrices viewed as $\mathbf{R}^{mn}$. Denote this norm by $|...|_2$.

Suppose that A is an $m \times n$ matrix representing T; it is enough to show that all matrices sufficiently close to A have rank n if (as is the case here) A has rank n. Since the rank of A is n, one can choose n rows from A to obtain an $n \times n$ matrix $\varphi(A)$ that is invertible. We know that the set of invertible square matrices is open, so there is a $\delta > 0$ such that $|C - \varphi(A)|_2 < \delta$ implies that C is invertible. Since $|\varphi(B) - \varphi(A)|_2 \leq |B - A|_2$ this implies that the rank of B is at least n if $|B - A|_2 < \delta$. Since the rank of B is at most n, it follows that the rank is exactly n under the given condition and hence that the linear transformation associated to B is 1–1.■

11. Let $1 \leq r \leq \infty$.

(*i*) If U is open in $\mathbf{R}^n$, prove that the identity map id_U is a $\mathcal{C}^\infty$ diffeomorphism.

SOLUTION.

The identity on U is a smooth map and equal to its own inverse. Therefore by the definitions it is a diffeomorphism.■

(*ii*) If U and V are open in $\mathbf{R}^n$ and $f : U \to V$ is a $\mathcal{C}^r$-diffeomorphism, then so is f^{-1}.

SOLUTION.

By hypothesis f^{-1} is a smooth map, and an inverse is just the original map f, which by hypothesis we also know is smooth.■

(*iii*) If U, V and W are open in $\mathbf{R}^n$, and $f : U \to V$ and $g : V \to W$ are $\mathcal{C}^r$ diffeomorphisms, then so is $g \circ f$.

SOLUTION.

The composite of smooth maps is smooth, so this implies $g \circ f$ is smooth. Also, an inverse to this map is given by $f^{-1} \circ g^{-1}$, which by hypothesis is also smooth. Therefore $(g \circ f)^{-1}$ is smooth, and by definition this means $g \circ f$ is a diffeomorphism.■

12. (*i*) Suppose that X and Y are subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively and that $f : X \to Y$ and $g : Y \to \mathbf{R}^p$ are maps that satisfy Lipschitz conditions. Prove that the composite $g \circ f$ also satisfies a Lipschitz condition. Prove this or give a counterexample.

SOLUTION.

By our assumptions there are positive constants A and B such that $|f(x) - f(x')| \leq A \cdot |x - x'|$ and $|g(y) - g(y')| \leq B \cdot |y - y'|$. If we set $y = f(x)$ and $y' = f(x')$ in these formulas we obtain

$$|g \circ f(x) - g \circ f(x')| \quad \leq \quad B \cdot |f(x) - f(x')| \quad \leq \quad AB \cdot |x - x'|$$

which shows that $g \circ f$ satisfies a Lipschitz condition.■

(*ii*) Suppose that $X \subset \mathbf{R}^n$, and let f, $g : X \to \mathbf{R}^m$ and $h : X \to \mathbf{R}$ satisfy Lipschitz conditions. Prove that $f + g$ satisfies a Lipschitz condition and if X is compact then $h \cdot f$ also satisfies a Lipschitz condition. If $h > 0$ and X is compact, does $1/h$ satisfy a Lipschitz condition? Prove this or give a counterexample.

SOLUTION.

Let A and B be Lipschitz constants for f and g respectively. Furthermore, let C be a Lipschitz constant for h.

We verify first that $f+g$ satisfies a Lipschitz condition:

$$|[f+g](x) - [f+g](x')| = |f(x)+g(x)-f(x')-g(x')| \ \le\ |f(x)-f(x')| + |g(x)-g(x')| \ \le$$

$$A\cdot|x-x'| + B\cdot|x-x'| \quad = \quad (A+B)\,|x-x'|$$

In order to prove that $h\cdot f$ satisfies a Lipschitz condition we need to use the upper bounds for $|f|$ and $|h|$ which are guaranteed by compactness. Call these bounds P and Q respectively. We then have that

$$|h(x)f(x) - h(x')f(x')| \quad = \quad |h(x)f(x) - h(x)f(x') + h(x)f(x') - h(x')f(x')| \ \le$$

$$|h(x)f(x) - h(x)f(x')| + |h(x)f(x') - h(x')f(x')| \ \le\ Q\cdot|f(x)-f(x')| + P\cdot|h(x)-h(x')| \ \le$$

$$QA\,|x-x'| + PC\,|x-x'| \quad = \quad (QA+PC)\,|x-x'|$$

and hence the product satisfies a Lipschitz condition.

Finally, $1/h$ does satisfy a Lipschitz condition, and here is the proof: In addition to the preceding let $\mathbf{m}$ be the minimum value of $|h|$ on X. Then we have

$$\left|\frac{1}{h(x)} - \frac{1}{h(x')}\right| =$$

$$\frac{|h(x') - h(x)|}{|h(x)\cdot h(x')|}\ .$$

The denominator is at least $\mathbf{m}^2$, and the numerator is at most $C\cdot|x-x'|$, and therefore $C\mathbf{m}^{-2}$ is a Lipschitz constant for $1/h$.■

(iii) Suppose that $X \subset \mathbf{R}^n$, and let $f : X \to \mathbf{R}^m$ be given. Prove that f satisfies a Lipschitz condition if and only if all of its coordinate functions do.

SOLUTION.

($\Longrightarrow$) If f satisfies a Lipschitz condition with Lipschitz constant A and f_i is the $i^{\rm th}$ coordinate function, then

$$|f_i(x) - f_i(y)| \ \le\ |f(x) - f(y)| \ \le\ A\cdot|x-y|$$

for all x and y.■

($\Longleftarrow$) If each coordinate function f_i satisfies a Lipschitz condition, then one can write $f = \sum_i f_i\,\mathbf{e}_i$ where $\mathbf{e}_i$ is the standard $i^{\rm th}$ unit vector; therefore the conclusion follows from the first part of the exercise and finite induction.■

13. In the notation of the preceding exercise, suppose that $X = A\cup B$ and that f is continuous and satisfies Lipschitz conditions on A and B as well as on an open neighborhood of $A\cap B$. Does f satisfy a Lipschitz condition on $A\cup B$? Prove this or give a counterexample. What happens if we assume A and B are compact? Justify your answer.

SOLUTION.

The answer to the first question is no. Consider the function on $\mathbf{R} - \{0\}$ that is 1 for positive numbers and -1 for negative numbers. This satisfies a Lipschitz condition on A and B as well as an open neighborhood of $A \cap B = \emptyset$. However, if we take x and x' to be $\pm 1/n$ then $|f(x) - f(x')| = 2$ while $|x - x'| = 2/n$, and hence any Lipschitz constant for f on $A \cup B$ would have to be at least n for every positive integer n. Therefore f does not satisfy a Lipschitz condition on $A \cup B$.

The answer to the second question is yes. Let U be an open neighborhood of $A \cap B$ on which f satisfies a Lipschitz condition, and let K_0 be the associated Lipschitz constant. Similarly, let K_1 and K_2 be Lipschitz constants for f on A and B respectively.

Formally, there are initially 64 cases to consider depending upon whether or not u lies in A, B or U (a total of 8 possibilities), and likewise for v. However, many of the formal possibilities are inconsistent with the conditions that u and v belong to $A \cup B$. In particular, we cannot have $u \notin A$ and $u \notin B$ and we cannot have $u \in A$, $u \in B$, but $u \notin U \supset A \cap B$. This brings us down to 5 possibilities each for u and v. Here is the list of possibilities for u:

[u1] $u \notin A$, $u \in B$ and $u \notin U$
[u2] $u \notin A$, $u \in B$ and $u \in U$
[u3] $u \in A$, $u \notin B$ and $u \notin U$
[u4] $u \in A$, $u \notin B$ and $u \in U$
[u5] $u \in A$, $u \in B$ and $u \in U$

Of course, there is a similar list for v, and the set of all 25 possibilities is given by taking one from each list.

Fortunately, separate arguments are not needed for each of the 25 cases. In particular, the four cases obtained by combining the first two possibilities for u and v involve situations where both points lie in B and therefore $|f(u) - f(v)| \leq K_2 |u - v|$. Likewise, the nine cases obtained by combining the last three possibilities for u and v involve situations where both points lie in A and therefore $|f(u) - f(v)| \leq K_1 |u - v|$.

This leaves us with twelve cases; six are given by taking one of the first two possibilities for u and one of the last three possibilities for v, and six are given by switching the roles of u and v. If we do the latter, we are down to six cases.

The cases $[u2] + [v2]$ and $[u2] + [v3]$ are situations where both point belong to U and therefore $|f(u) - f(v)| \leq K_0 |u - v|$. In each of the remaining cases, at least one of u or v does not belong to U. Consider the continuous function on

$$\big((A \cup B) \times (A \cup B)\big) - U \times U$$

defined by the quotient

$$\frac{|f(u) - f(v)|}{|u - v|} \;.$$

Since the domain of this function is compact, it has a maximum value K_3, and thus we have $|f(u) - f(v)| \leq K_3 |u - v|$ if (u, v) lies in the set described above.

If we take K to be the largest of the numbers K_i for $0 \leq i \leq 3$, then K will be a Lipschitz constant for f on $A \cup B$.∎

II.2 : Implicit and Inverse Function Theorems

(Conlon, Appendix B, §§ 2.4–2.5)

Additional exercises

1. Show that

$$f(x,y) \quad = \quad \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$$

is locally invertible near every point except the origin. Compute the inverse explicitly.

SOLUTION.

Since the problem asks for an explicit computation of the inverse, one reasonable way to start is to see what happens if one tries to solve the system of equations given in vector form by

$$(u,v) \quad = \quad \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$$

(this is easier than writing out two separate equations for u and v in terms of x and y). Next, think about what this map does geometrically. Given a nonzero point in the plane, it sends this point to a positive multiple of itself; if as usual we write $r^2 = x^2 + y^2$, the exact multiple is $1/r^2$. Formally one can see this by verifying the identity

$$u^2 + v^2 \quad = \quad \frac{1}{x^2+y^2} \ .$$

Using this formula it follows immediately that f maps the nonzero points of the plane to themselves in a 1–1 onto fashion and is equal to its own inverse. Since f is a $\mathbf{C}^1$ function, it follows in particular that the function is *globally* invertible on the set of nonzero points of the plane.■

2. Consider the map : $\mathbf{R}^3 \to \mathbf{R}^3$ defined by $f(x,y,z) = (x, y^3, z^5)$. Note that f has a global inverse g despite the fact that $Df(0)$ is not invertible. What does this imply about the differentiability of g at 0?

SOLUTION.

The map cannot be differentiable at the origin, for if it were then it would be an inverse to $Df(0)$ and the latter does not have an inverse. Of course the global continuous inverse to f is the function $g(u,v,w) = (u, v^{1/3}, w^{1/5})$.■

3. Show that the mapping $(u,v,w) : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $u = x + e^y$, $v = y + e^z$ and $w = z + e^x$ is everywhere locally invertible.

SOLUTION.

Compute the Jacobian:

$$\begin{vmatrix} 1 & e^y & 0 \\ 0 & 1 & e^z \\ e^x & 0 & 1 \end{vmatrix} \quad = \quad 1 + e^{x+y+z} \quad \neq \quad 0 \ .■$$

4. Let $f : \mathbf{R}^3_{\mathbf{x}} \to \mathbf{R}^3_{\mathbf{y}}$ and $g : \mathbf{R}^3_{\mathbf{y}} \to \mathbf{R}^3_{\mathbf{x}}$ be $\mathcal{C}^1$ inverse functions. Show that

$$\frac{\partial g_1}{\partial y_1} \quad = \quad \frac{1}{J} \frac{\partial(f_2, f_3)}{\partial(x_2, x_3)}, \qquad J = \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)}$$

and obtain similar formulas for the other derivatives of coordinate functions of g.

SOLUTION.

By construction Dg and Df are 3×3 matrices that are inverse to each other. By Cramer's Rule, if B and A are mutually inverse 3×3 matrices, then

$$b_{1,1} = \frac{1}{\det A} \cdot \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix}$$

and the formula for the partial derivative as a quotient ot two Jacobians follows immediately from this. Similar considerations hold for all the partial derivatives of coordinate functions; since writing things out would make the solution about eight times longer and the details are mechanical, we shall not do so.■

5. Prove that $F(x, y) = (e^x + y,\ x - y)$ defines a $\mathcal{C}^\infty$ homeomorphism of $\mathbf{R}^2$ with a $\mathcal{C}^\infty$ inverse.

SOLUTION.

By the Inverse Function Theorem it suffices to show that F is 1–1 onto $\mathbf{C}^\infty$ and has a nonzero Jacobian everywhere. The $\mathbf{C}^\infty$ condition is immediate, and one can compute the Jacobian directly:

$$\begin{vmatrix} e^x & 1 \\ 1 & -1 \end{vmatrix} \quad = \quad -e^x - 1 \quad < \quad 0$$

To show that f is 1–1 onto we need to show that for each choice of u and v there is a unique solution to the system of equations

$$u \quad = \quad e^x + y, \qquad\qquad v \quad = \quad x - y\ .$$

Here is one way of doing so. If we add the two equations we find that $x + e^x = u + v$. But the function $h(x) = x + e^x$ is a strictly increasing function whose limits at $\pm\infty$ are $\pm\infty$ respectively, and therefore there is a unique inverse function $k(x) : \mathbf{R} \to \mathbf{R}$ that is infinitely differentiable. It follows that $x = k(u+v)$. Applying this to the second equation, we obtain the relation $y = x - v = k(u+v) - v$. Therefore F is 1–1 onto, and it is also $\mathbf{C}^\infty$ with everywhere nonvanishing Jacobian as required.■

6. Prove that $F(x, y) = (xe^y + y,\ xe^y - y)$ defines a $\mathcal{C}^\infty$ homeomorphism of $\mathbf{R}^2$ with a $\mathcal{C}^\infty$ inverse.

SOLUTION.

By the Inverse Function Theorem it suffices to show that F is 1–1 onto $\mathbf{C}^\infty$ and has a nonzero Jacobian everywhere. The $\mathbf{C}^\infty$ condition is immediate, and one can compute the Jacobian directly:

$$\begin{vmatrix} e^y & x\,e^y + 1 \\ e^y & x\,e^y - 1 \end{vmatrix} \quad = \quad -2e^y \quad \neq \quad 0$$

To show that f is 1–1 onto we need to show that for each choice of u and v there is a unique solution to the system of equations

$$u \quad = \quad xe^y + y, \qquad\qquad v \quad = \quad xe^y - y\ .$$

Here is an elementary way of doing so. Subtacting the second equation from the first shows that $y = (u-v)/2$, and adding the two equations together yields

$$2\,x\,e^{y} \quad = \quad u + v \ .$$

Since we can solve uniquely for y, this equation shows that we can also solve uniquely for x. Therefore F is 1–1 onto, and it is also $\mathbf{C}^{\infty}$ with everywhere nonvanishing Jacobian as required.■

7. Prove that

$$F(x,y,z) \quad = \quad \left(\frac{x}{2+y^2} + ye^{z}, \frac{x}{2+y^2} - ye^{z}, 2ye^{z} + z \right)$$

defines a $\mathcal{C}^{\infty}$ homeomorphism of $\mathbf{R}^3$ with a $\mathcal{C}^{\infty}$ inverse.

SOLUTION.

Use the same approach as in the previous problem. The map is $\mathbf{C}^{\infty}$ by construction, and its Jacobian is the following 3×3 determinant:

$$\left| \begin{array}{ccc} 1/(2+y^2) & -2\,xy/(2+y^2)^2 + e^{z} & ye^{z} \\ 1/(2+y^2) & -2\,xy/(2+y^2)^2 - e^{z} & -ye^{z} \\ 0 & 2e^{z} & 2ye^{z} + 1 \end{array} \right|$$

One can use row and column operations to simplify the computation before writing everything out algebraically, but in any case the Jacobian is equal to

$$\frac{-2\,e^{z}}{2+y^2} \quad < \quad 0 \ .$$

The next step is to show that one can find unique solutions to the system of equations

$$(u,v,w) \quad = \quad \left(\frac{x}{2+y^2} + ye^{z}, \frac{x}{2+y^2} - ye^{z}, 2ye^{z} + z \right)$$

and here is a summary of how this can be done: Subtracting the second equation from the first yields $2\,y\,e^{z} = u - v$, and by the third equation the left hand side is equal to $w - z$. Therefore we can solve for z uniquely in terms of u, v and w. If we substitute this result into $2\,y\,e^{z} = u - v$ we also get a unique solution for y in terms of u, v and w. Finally, if we add the original first and second equations we obtain $u + v = 2\,x/(2+y^2)$. Since we already know that we can solve uniquely for y, this equation implies that we also get a unique solution for x terms of u, v and w.■

8. Let $f(x,y) = (x+y, x^2+y)$. Check that f meets the conditions to have a local inverse near $f(1,0) = (1,1)$, and if g is this local inverse find $Dg(1,1)$ without finding a formula for the inverse function explicitly.

SOLUTION.

One again begin by writing down the Jacobian:

$$\left| \begin{array}{cc} 1 & 1 \\ 2\,x & 1 \end{array} \right| \quad = \quad 1 - 2x$$

The right hand side is nonzero at $(1,0)$ and the derivative matrix $Df(1,0)$ is equal to

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} .$$

By the Chain Rule, $Dg(1,1) = [Df(1,0)]^{-1}$, and by either Cramer's rule or one's favorite matrix inversion technique the latter is equal to

$$\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} .\blacksquare$$

9. Consider the mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$ given by $f(x,y) = (x^2+y^2, 2xy)$. Show that the Jacobian vanishes on the lines $y = \pm x$. What is the image of f? [*Hint:* Try using polar coordinates.] The Inverse Function Theorem guarantees that f has a local inverse at $f(1,0) = (1,0)$. Find the inverse explicitly and describe a region on which it is defined.

SOLUTION.

Once again, compute the Jacobian:

$$\begin{vmatrix} 2x & 2y \\ 2y & 2x \end{vmatrix} \quad = \quad 2(x^2 - y^2)$$

This clearly vanishes on the lines $y = \pm x$. To find the image, use the hint to rewrite $f(x,y) = g[r,\theta] = (r^2, r^2 \sin 2\theta)$, where parentheses refer to rectangular coordinates and square brackets refer to polar coordinates. The polar expressions tell us that the image consists of all points (u,v) for which $u \geq 0$ and $|v| \leq u$.

To find the local inverse explicitly write $u = x^2 + y^2$ and $v = 2xy$. Solving this system of simultaneous quadratic equations is essentially an exercise in high school algebra. Since we are solving near $(1,0)$ we may divide by x more or less freely. It turns out that the solution for x and y in terms of u and v such that $(x,y) = (0,1)$ when $(u,v) = (0,1)$ is given by

$$x = \sqrt{\frac{u + \sqrt{u^2 - v^2}}{2}}, \quad y = \frac{v}{2x} .$$

We have not expressed x explicitly in terms of u and v in the second equation, but substitution of the first equation into the second enables one to write y explicitly in terms of u and v. This formula is valid for all (u,v) such that $u > 0$ and $|v| < u$.$\blacksquare$

10. The following example shows why it is necessary to assume the continuity at a point in the Inverse Function Theorem. Let $f(t) = t + 2t^2 \sin\left(\frac{1}{t}\right)$ for $t \neq 0$ and set $f(0) = 0$. Prove that $f'(0) = 1$, f' is bounded on $(-1,1)$, but f is not 1–1 on any neighborhood of 0.

SOLUTION.

The phrase "continuity at a point" means continuity of the derivative at a point.

By the additivity of derivatives, the first statement will be true if we can show that the function $g(t) = 2t^2 \sin\left(\frac{1}{t}\right)$ satisfies $g'(0) = 0$. Therefore we must consider

$$\lim_{h \to 0} 2h \sin\left(\frac{1}{h}\right) .$$

But the expression on the right always lies between $\pm 2\,h$ and the limits of these functions as $h \to 0$ is zero, so the limit which defines $g'(0)$ must also exist and be equal to zero.

For similar reasons, proving the boundednes of f' is also equivalent to proving the boundedness of g'. If $t \neq 0$ this derivative is equal to

$$4t \sin\left(\tfrac{1}{t}\right) \;-\; 2 \cos\left(\tfrac{1}{t}\right)$$

an expression whose absolute value is clearly ≤ 6 on the interval $(0, 1)$.

Finally, f cannot be 1–1 on any neighborhood of 0 for the following reason. If it were differentiable and 1–1 on some interval $(-\delta, \delta)$, then the sign of the derivative would be either nonpositive or nonnegative on the whole interval. Thus it suffices to prove that the derivative is both positive and negative on an arbitrary small open neighborhood of 0. But $f'(t) > 0$ if $t = 4/(4k\,\pi + 1)$ and $f'(t) < 0$ if $t = 1/(2k\,\pi + 1)$, so therefore f cannot be 1–1 on any neighborhood of zero.■

11. (*i*) Let W be open in $\mathbf{R}^n$, and let $h : W \to \mathbf{R}^k$ be continuous. Prove that h is smooth if and only if there is an open covering $\mathcal{V}$ of W such that for each V_α in $\mathcal{V}$ the restriction $f|V_\alpha$ is smooth.

SOLUTION.

Smoothness at a point is defined in terms of the function's behavior in a small open neighborhood of each point, and this neighborhood can be chosen to be arbitrarily small. The hypothesis implies that for each point one can find an open neighborhood on which the smoothness criterion is satisfied. Therefore f is smooth at every point of W.■

(*ii*) Let U and V be open in $\mathbf{R}^n$, let $f : U \to V$ be a smooth surjective immersion/submersion, and suppose that $g : V \to \mathbf{R}^q$ is a continuous map such that $g \circ f$ is smooth. Prove that g is also smooth.

SOLUTION.

One should add an assumption that f is onto; it is possible to construct counterexamples otherwise (think about a function g that is smooth on an open subset $U \subset V$ but not smooth at some point in $V - U$, and take f to be the inclusion map.

Assuming f is onto, given $x \in V$ choose $y \in U$ such that $f(y) = x$. Then there are open neighbhooods U_0 of y and V_0 of x such that f defines a diffeomorphism f_0 from U_0 to V_0. Let h be its inverse. Then $g|V_0 = (f|U_0) \circ h$ and this factorization implies that $g|V_0$ is smooth. If we do this for each $x \in V$ we obtain an open covering of V by sets V_x such that $g|V_x$ is smooth for all x. We may now apply (*i*) to conclude that g is smooth.■

12. A continuous map $f : A \to X$ is a *retract* if there is a continuous map $g : X \to A$ such that $g \circ f = \mathrm{id}_A$. Suppose that A and X are open subsets of Euclidean spaces and f and g are smooth. Prove that f is an immersion.

SOLUTION.

The derivative $D\mathrm{id}_A$ is just the identity linear transformation on the ambient Euclidean space, and therefore by the Chain Rule we have

$$I \;=\; Dg(f(x)) \cdot Df(x)$$

for all $x \in A$. Now if we are given any pair of matrices B and C such that $C\,B = I$, then it follows that the the null space of B is the zero space, for $B\,v = 0 \Longrightarrow 0 = I \cdot 0 = C\,0 = C\,B\,v = I\,v = v$. Taking $B = Df(x)$, we conclude that the latter is always 1–1 and therefore f is an immersion.■

II.3 : Bump functions

(Conlon, §2.6)

Additional exercises

1. Let $f : \mathbf{R} \to \mathbf{R}$ be a smooth function of class $\mathcal{C}^r$ where $1 \leq r \leq \infty$, and let K be a compact subset of $\mathbf{R}$. Then there is a smooth $\mathcal{C}^r$ function $g : \mathbf{R} \to \mathbf{R}$ such that $g|K = f|K$ and g vanishes off some compact set K' containing K. [*Hint:* We can take $K = [-a, a]$ and $K' = [-b, b]$ for some a, b such that $0 < a < b$.]

SOLUTION.

Following the hint, we first show that we might as well assume K and K' are intervals as described there. If L is an arbitrary compact set, then since it is bounded it lies in some compact set $K = [-a, a]$. If we can find a g such that $g|K = f|K$, then automatically we also have $g|L = f|L$. Furthermore if the function g vanishes off some interval $[-b, b]$ containing K, then it still vanishes off some compact set containing L.

All we need to do now is take a bump function φ that is 1 on $[-a, a]$ and 0 on $[-b, b]$, and take g to be the product of f and φ.■

2. Suppose that $A \subset \mathbf{R}^n$ and $f : A \to \mathbf{R}^n$ is continuous. Suppose further that for each $a \in A$ there is an open neighborhood V_a of a such that $f|A \cap V_a$ extends to a smmoth function on V_a. Prove that there is an open set W containing A and a smooth function $g : W \to \mathbf{R}$ such that $g|A = f$. [*Hint:* Start with a locally refinement $\mathcal{U}$ of $\mathcal{V} = \{V_a\}$ and a partition of unity subordinate to $\mathcal{U}$.]

SOLUTION.

Let g_a be a smooth function on V_a which agrees with f on $V_a \cap A$, and for each U_α choose V_a such that $U_\alpha \subset V_a$. Let h_α be the restriction of g_α to U_α, and let $\{\varphi_\alpha\}$ be a smooth partition of unity subordinate to $\mathcal{U}$. Then each function $\varphi_\alpha \cdot h_\alpha$ extends smoothly to $\mathbf{R}^n$ by setting it equal to zero outside U_α (as usual, the function vanishes off a compact subset of U_α and this suffices to guarantee continuity). If we take $g = \sum_\alpha \varphi_\alpha \cdot h_\alpha$ then g is a smooth function defined on a neighborhood of A and $x \in A$ implies

$$g(x) \;=\; \sum_\alpha \varphi_\alpha(x) \cdot h_\alpha(x) \;=\; \sum_\alpha \varphi_\alpha(x) \cdot f(x) \;=$$

$$f(x) \cdot \left(\sum_\alpha \varphi_\alpha(x) \right) \;=\; f(x) \cdot 1 \;=\; f(x)$$

so that $g|A = f$ as required.■

3. The following exercise will be based upon an important result for uniform convergence of infinite series to a differentiable function that follows from a more general result: Theorem 7.17 on pp.152–153 of BABY RUDIN: *Suppose we are given a sequence of uniformly absolutely* *convergent smooth $\mathcal{C}^1$ functions $\{ f_n \}$ on an interval U such that $\sum_n f_n'$ also converges uniformly and absolutely. Then f is a smooth function and $f' = \sum_n f_n'$.*

(*i*) Explain why this result generalizes to smooth $\mathcal{C}^1$ functions defined on an open set $U \subset \mathbf{R}^q$ for some $q \geq 0$ with f_n' replaced by ∇f_n (and vector length replacing the absolute value of a real number).

SOLUTION.

We can apply the same method to show that the limit function f has continuous partial derivatives in all directions and that

$$\frac{\partial f}{\partial x_j} \quad = \quad \lim_{n \to \infty} \frac{\partial f_n}{\partial x_j}$$

for each j. But this means that f has continuous partials and hence f is a smooth $\mathcal{C}^1$ function.∎

(*ii*) Let $U \subset \mathbf{R}^q$ be open, and let F be a closed subset of U. Prove that there is a smooth $\mathcal{C}^1$ function $h : U \to \mathbf{R}$ such that for all $x \in U$ we have $h(x) = 0 \Longleftrightarrow x \in F$; *i.e.*, in analogy with a result about continuous functions on metric spaces, every closed subset of U is the zero set for some smooth $\mathcal{C}^1$ function on U. [*Hints:* Take the usual sort of locally finite countable open covering of $U - F$ by ordinary open disks such that shrunken disks of half the radius still cover $U - F$, and let g_k be the smooth function defined on the k^{th} disk using a bump function, where as usual g_k extends smoothly to all of U by setting it equal to zero off the disk. Choose positive constants M_k such that $|g_k|$ and $|\nabla g_k|$ are both bounded from above by M_k, and set

$$h \quad = \quad \sum_k \frac{g_k}{M_k \cdot 2^k} \ .$$

Explain why h is a smooth $\mathcal{C}^1$ function and the zero set of h is equal to F.]

SOLUTION.

If we follow the hint it is only necessary to show the claims in its final sentence. To see that the zero set is equal to F, note that $x \notin F$ implies that some bump function $g_k(x)$ is nonzero, and hence the sum of all the nonnegative numbers $g_k(x)$ must be positive. By the observations of the first part of this exercise, this infinite sum will be a smooth $\mathcal{C}^1$ function if the sum

$$\mathbf{G} \quad = \quad \sum_k \left(\frac{1}{M_k \cdot 2^k} \right) \cdot \nabla g_k$$

converges absolutely. However, by construction we know that

$$\sum_k \left(\frac{1}{M_k \cdot 2^k} \right) \cdot |\, \nabla g_k \,| \quad \leq \quad \sum_k \frac{1}{2^k} \quad = \quad 2$$

and this implies the uniform convergence of $\mathbf{G}$.∎

II.4 : Vector fields and integral flows

(Conlon, §§2.7–2.8, Appendix C.1–C.3)

Additional exercises

1. Find the flow associated to the vector field on $\mathbf{R}^2$ given by

$$y\frac{\partial}{\partial x} - y^3\frac{\partial}{\partial y}.$$

SOLUTION.

In the language of undergraduate differential equations courses, this translates to the system $x' = y$, $y' = -y^3$. The second equation is separable and the first is obtainable by finding an "indefinite integral." However, a little care will be needed to write out the general solutions for reasons that will become apparent during the process of finding solutions.

Let's start with the second equation. The standard procedure for finding y begins by considering

$$-\frac{1}{y^3}\frac{dy}{dt} \quad = \quad 1$$

which leads to the solution $1/y^2 = 2\,(C-t)$ where C is a constant of integration. Taking square roots we find that $y = \pm\, 1/\sqrt{2\,(C-t)}$. Note that the manipulations leading to this solution tacitly assumed $y \neq 0$ in order to rewrite the equation in the displayed form. Fortunately, this is only a minor problem because the unique solution to $y' = y^3$ with initial condition $y(0) = 0$ is the zero function; we need to remember this when it is time to write down general formulas.

We now want to write the solutions to $y' = -y^3$ in terms of t and the initial condition $v = y(0)$. This means we must solve for the constant of integration C in terms of v. By direct substitution we know that

$$v \quad = \quad \pm\sqrt{2\,(C)}$$

and this implies $C = 1/(2\,v^2)$. Note that this requires the initial condition v to be nonzero. If we substitute this value for C and rewrite the solutions slightly we obtain the following expression for the solution:

$$y(t) \quad = \quad \frac{\pm\sqrt{2\,v^2}}{\sqrt{2 - 2\,v^2\,t}}$$

Although this formula was obtained for the case $v \neq 0$, it also works if $v = 0$ for trivial reasons (as noted before, the solution then is identically zero). The only remaining problem is to determine the choice of sign. However, this is straightforward by the relation $y(0) = v$; specifically, we must choose the signs so that $\pm\sqrt{v^2} = v$. Therefore the second component of the integral flow map $\Phi(t; u, v)$ is given by

$$y(t) \quad = \quad \frac{v\,\sqrt{2}}{\sqrt{2 - 2\,v^2\,t}}$$

and this works even if $v = 0$.

We must now describe the first coordinate using the relation $x' = y$. One way to start is to compute an indefinite integral:

$$x(t) \quad = \quad \int \frac{\pm\,dt}{\sqrt{2\,(C-t)}} \quad = \quad \mp\sqrt{2\,(C-t)} \; + \; B \quad = \quad -\frac{1}{y(t)} \; + \; B$$

Here B is a second constant of integration and once again we are assuming that $y \neq 0$ (we already know what happens if $y = 0$; in this case the value of the vector field is 0 and the solution is a constant curve). We may solve for the constant of integration B in terms of the initial condition $u = y(0)$ exactly as before to obtain the formula

$$B \quad = \quad u \, + \, \frac{1}{v} \; .$$

This implies that the first coordinate of the flow map $\Phi(t; u, v)$ is given by the following formula:

$$x(t) \quad = \quad u \, + \, \frac{1}{v} \quad - \quad \frac{v\sqrt{2}}{\sqrt{2 - 2\,v^2\,t}}$$

If we combine all these obseervations we conclude that $\Phi(t; u, v) = (\, x(t), \; y(t)\,)$ where $x(t)$ and $y(t)$ are given as above.■

2. Find the flow associated to the vector field on $\mathbf{R}^3$ given by

$$ay\frac{\partial}{\partial x} - ax\frac{\partial}{\partial y} + a^2\frac{\partial}{\partial z}.$$

SOLUTION.

In this case the system one obtains is $x' = a\,y, \;\; y' = -a\,x$, and $z' = -a^2$. The third equation is independent of the others, and the general solution is $z = -a^2\,t + K$ where K is a constant of integration. The other two yield the second order differential equation $x'' = -a^2\,x$, whose general solution is $B\,\cos\,at + C\,\sin\,at$ where B and C are constants of integration. We can then substitue to find that $y = -B\,\sin\,at + C\,\cos\,at$.

As in the previous exercise, in order to give a formula for the flow we need to solve for the constants of integration in terms of $u = x(0)$, $v = y(0)$ and $w = z(0)$. Elementary substitution shows that $(B, C, K) = (u, v, w)$, and hence the flow for this equation is given by

$$\Phi(t; u, v, w) \quad = \quad (u\,\cos\,at \; + \; v\,\sin\,at, \; -u\,\sin\,at \; + \; v\,\cos\,at, \; w \; - \; a^2\,t)\;.■$$

3. Find the flow associated to the vector field on $\mathbf{R}^3$ given by

$$y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

SOLUTION.

Straightforward manipulation of the equations in this system leads to the equations $x'' = z$ and $x''' = x$. The latter is an ordinary third order homogeneous equation with constant coefficients that we can solve by standard undergraduate methods. The latter show that the general solution has the form

$$x \quad = \quad A\,e^t \; + \; B\,\exp(-t/2)\,\cos(\tfrac{1}{2}t\sqrt{3}) \; + \; C\,\exp(-t/2)\,\sin(\tfrac{1}{2}t\sqrt{3})$$

where A, B, C are constants of integration. We can relate them to the initial values u, v, w by the identities $u = x(0)$, $v = y(0) = x'(0)$ and $w = z(0) = x''(0)$. This yields three equations in the constants of integration:

$$u = A + B + C$$

$$v = A - \tfrac{1}{2}B + \tfrac{1}{2}\sqrt{3}\cdot C$$

$$w = A - \tfrac{1}{2}B - \left(1 + \tfrac{1}{2}\sqrt{3}\right)\cdot C$$

Instead of writing out everything explicitly, we shall simply indicate how one retrieves the general solution from these data. Solving this system for the constants of integration in terms of u, v, w yields a formula for $x(t)$ in terms of the latter; this is the first coordinate of the flow mapping $\Phi(t, u, v, w)$. The second coordinate may then be obtained by differentiating $x(t)$ and using the formula $x' = y$, and the third coordinate may be found similarly by differentiating $y(t)$ and using the formula $z = y'$.■

4. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation that has a basis of eigenvectors $\{\mathbf{v}_j\}$ with associated eigenvalues λ_j. Given a vector $\mathbf{x} \in \mathbf{R}^n$, express $\mathbf{x}$ as a linear combination $\sum_j c_j \mathbf{x}_j$. Verify that

$$\gamma(t) = \sum_j c_j \exp(\lambda_j t)\, \mathbf{v}_j$$

is a solution to the differential equation $\mathbf{y}' = T(\mathbf{y})$ with initial condition $\mathbf{x}(0) = \mathbf{x}$.

SOLUTION.

Direct computation yields the following formula for $\gamma'(t)$:

$$\gamma'(t) = \sum_j c_j \lambda_j \exp(\lambda_j t)\, \mathbf{v}_j$$

This is exactly the same expression that one gets by evaluating $T \circ \gamma(t)$. Checking the initial condition we find that

$$\gamma(0) = \sum_j c_j \exp(0 \cdot t)\, \mathbf{x}_j = \sum_j c_j \mathbf{x}_j = \mathbf{x}$$

and hence the initial conditiion is $\mathbf{x}(0) = \mathbf{x}$ as required.■

5. Show that the differential equation $y' = y^{2/3}$ with initial condition $y(0) = 0$ has infinitely many solutions. [*Hint:* Consider the functions y such that $y(t) = 0$ for $t \leq a$ and $y(t) = (t-a)^3$ for $t \geq a$. Some care is needed to compute the derivative of this function at $t = a$.]

SOLUTION.

First of all, the hint should be corrected as above, so that we are considering functions y_a such that $y_a(t) = 0$ for $t \leq a$ and $y(t) = (t-a)^3$ for $t \geq a$.

Simple compuatations show that all the functions in the hint satisfy the differential equation when $t \neq a$. At $t = a$ the differential equation will be satisfied if and only if y_a is differentiable there and $y'(a) = 0$. To prove this it is necessary to consider each of the left and right hand limits separately:

$$\lim_{t \to a+} \frac{f(t) - f(a)}{t - a} \quad , \quad \lim_{t \to a-} \frac{f(t) - f(a)}{t - a}$$

The formulas for these quotients are different, but each of the limits is equal to zero.■

6. Here is a slightly different application of the Contraction Lemma to a boundary value problem in the theory of differential equations.

(*i*) Suppose that $F : [a,b] \times \mathbf{R} \to \mathbf{R}$ is Lipschitz and K is a Lipschitz constant for F. Define a *Green's function* $G : [a,b] \times [a,b] \to \mathbf{R}$ by setting

$$G(s,t) \quad = \quad \begin{cases} \frac{(t-a)(b-s)}{(b-a)} & t \le s \\ \frac{(s-a)(b-t)}{(b-a)} & s \le t\ . \end{cases}$$

Note that this function is discontinuous on the diagonal but still integrable. Verify that a continuous function y on $[a,b]$ satisfies $y(t) = \int_a^b G(t,s)\,F(s,\,y(s))\,ds$ if and only if it satisfies the boundary value problem $y'' + F(t,y) = 0$, $y(a) = y(b) = 0$.

SOLUTION.

Details for this entire exercise may be found on pages 188–193 of the following reference:

P. Waltman. **A Second Course in Elementary Differential Equations.** Academic Press, Orlando, FL, 1986. ISBN: 0–12–733910–8.

(*ii*) Show that $\int_a^b |G(t,s)|\,ds \le (b-a)^2/4$.

SOLUTION.

See (*i*).■

(*iii*) Show that if $b - a$ is so small that $K(b-a)^2/4 < 1$, then there is a unique solution to the boundargy value problem $y'' + F(t,y) = 0$, $y(a) = y(b) = 0$. [*Hint:* Define T by $T\varphi(t) = \int_a^b G(t,s)\,F(t,\,\varphi(t))\,ds$ and show that T satisfies the hypothesis of the Contraction Lemma.]

SOLUTION.

See (*i*) again. The same book outlines another boundary value problem that can be solved similarly (see Exercises 6–7 on pages 193–194).■