Math 205C
Spring 2003

Take home assignment 1

Due Wednesday, May 4, 2005

1. Let U be an open subset of $\mathbf{R}^n$, let $\mathcal{A}$ be a smooth atlas for U containing the standard chart (U, J), were J is the identity map on U, and let (U, h) be an arbitrary smooth chart in $\mathcal{A}$.

(i) How can one express the transition map "$J^{-1}h$" in terms of h?

(ii) Why does this imply that h is smooth?

2. Suppose that $f : M \to M'$ ahd $g : N \to N'$ are smooth maps. Prove that the map $f \times g : M \times N \to M' \times N'$ defined by $f \times g(x, y) = (f(x),\, g(y))$ is smooth. [*Hint* Let p_1 and p_2 be the projections from $M \times N$ to M and N respectively, and similarly let q_1 and q_2 be the projections from $M' \times N'$ to M' and N' respectively. Consider the composites of $f \times g$ with q_1 and q_2.]

3. Suppose that M is a topological 2-manifold. Prove that for each point $x \in M$ there is a neighborhood base $\{U_\alpha\}$ such that for each α we have

(i) The set $U_\alpha - \{x\}$ is connected, and the fundamental group of $U_\alpha - \{x\}$ with respect to some (in fact any) basepoint is nontrivial.

(ii) If $U_\beta \subset U_\alpha$ and $y \in U_\beta$ is different from x, then the includion of $U_\beta - \{x\}$ in $U_\alpha - \{x\}$ gives rise to an isomorphism of fundamental groups.

4. Suppose that M is a topological n-manifold for some $n \geq 3$.

(i) Prove that for each point $x \in M$ there is a neighborhood base $\{U_\alpha\}$ such that U_α is simply connected.

(ii) Explain why the conditions

M is a topological 1-manifold,

M is a topological 2-manifold,

M is a topological 3-manifold,

are mutually exclusive without using Brouwer's Invariance of Domain or Dimension theorems.