Mathematics 246A

Algebraic Topology — I

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Preface

This course is a continuation of the entry level graduate courses in algebraic topology given during the past two years (Mathematics 205C in Spring 2011 and Mathematics 205B in Winter 2012). In these courses we discussed an algebraic construction on spaces known as **singular homology theory**, which gives algebraic "pictures" of topological spaces in terms of certain abelian groups. We did not actually construct the theory, but we did the following:

(1) For a certain class of spaces known as *polyhedra*, we defined simplicial homology groups which turn out to be isomorphic to singular homology groups.

(2) We gave a somewhat lengthy list of properties or **axioms** for singular homology theory which turn out to characterize the theory uniquely up to natural isomorphism. The equivalence of simplicial homology groups with singular homology groups was included in this list of axioms.

This approach allowed us to use work with simplicial homology and use it to answer some easily stated topological and geometric problems, illustrating that homology theory is an effective tool for analyzing some fundamental types of questions in these subjects. However, *the answers derived in the earlier course(s) are contingent upon knowing that there actually is a singular homology theory satisfying the given axioms*. Thus the first goal of this course is to construct such a theory. In order to motivate the construction further, we shall also give a few applications beyond those in the previous entry level course; one possible example is a topological proof of the Fundamental Theorem of Algebra.

The approach described above can be compared to the way that one often studies the real number system, which is completely characterized by the algebraic and order-theoretic axioms for a *complete ordered field*. These axioms suffice to prove everything that one might want to prove in the theory of functions of real variables, but at some point it is necessary to show that there actually is a system which satisfies the axioms. This is generally done either by means of Dedekind cuts or equivalence classes of rational Cauchy sequences. In either case, once the constructions have yielded a complete ordered field, they have basically served their purpose and one does not need to remember the details of the construction.

The situation for singular homology theory is somewhat different, for one needs the details of the formal construction in order to refine the theory even further, and the next phase of the course will involve such refinements. In somewhat oversimplified terms, we can describe the situation as follows: When we think of algebra, we think of a system which has both addition and multiplication. Homology groups have an obvious additive structure, but in the previous course we did not really say anything about a multiplicative structure. It turns out that a very substantial multiplicative structure exists, and an understanding of the standard construction for singular homology is almost indispensable for motivating and working with this additional structure. We shall try to give applications of this extra structure to a few clearly basic mathematical problems whose statements do not involve homology.

The methods of algebraic topology turn out to be extremely effective for studying many sorts of questions involving topological or smooth manifolds, even in simple cases like open subsets of $\mathbb{R}^n$ (*i.e.*, questions in *geometric topology*), and the final portion of the course will be devoted to establishing a rew fundamental algebraic tools for studying such manifolds (the specifics depend upon time constraints). For example, one topic might be a unified approach to certain fundamental results in multivariable calculus involving the ∇ operator, Green's Theorem, Stokes' Theorem and the Divergence Theorem(s) in 2 and 3 dimensions, and to formulate analogs of these results for higher dimensions. A related topic could be the relationships among various approaches to defining an orientation for a manifold.

Course references

Mathematics 205A and 205B are prerequisites for this course. Lecture notes for these courses are available at the sites given below; the directories containing these files also contain exercises and other related documents (remove the `pdf` file names to get the links for the directories).

`http://math.ucr.edu/~res/math205A-2014/gentopnotes2014.pdf`

`http://math.ucr.edu/~res/math205A-2014/fundgp-notes.pdf`

`http://math.ucr.edu/~res/math205B-2012/algtop-notes.pdf`

Some topics near the end of the second document will be covered at the start of this course.

More formally, throughout the course we shall use the following texts for the basic graduate topology courses as references for many topics and definitions (the first and third are the current texts, and the second might be a helpful bridge between them):

J. R. Munkres. *Topology* (Second Edition), *Prentice-Hall, Saddle River NJ*, 2000. ISBN: 0–13–181629–2.

J. M. Lee. *Introduction to Topological Manifolds* (Second Edition), *Springer-Verlag, New York*, 2010. ISBN: 1–441–97939–5.

J. M. Lee. *Introduction to Smooth Manifolds*, *Springer-Verlag, New York*, 2002. ISBN: 0–387–95448–6.

The official text for this course is the following book:

A. Hatcher. *Algebraic Topology* (Third Paperback Printing), *Cambridge University Press, New York NY*, 2002. ISBN: 0–521–79540–0.

This book can be legally downloaded from the Internet at no cost for personal use, and here is the link to the online version:

`www.math.cornell.edu/~hatcher/AT/ATpage.html`

This web page also contains links to numerous updates, including corrections (one might add that solutions to many exercises are posted online and fairly easy to find using Google or something similar).

Comments on Hatcher's book. This text covers far more material than can be covered in two quarters, and in fact one could easily spend four quarters or three semesters covering the topics in that book by inserting a few extra topics. The challenges faced in covering so much ground are formidable. In particular, the gap between abstract formalism and geometrical intuition is significant, and it is not clear how well any single book can reconcile these complementary factors.

More often than not, algebraic topology books stress the former at the expense of the latter, and one important strength of Hatcher's book is that its emphasis tilts very much in the opposite direction. The book makes a sustained effort to include examples that will provide insight and motivation, using pictures as well as words, and it also attempts to explain how working mathematicians view the subject. Because of these objectives, the exposition in Hatcher is significantly more casual than in most if not all other books on the subject. Online reviews suggest that many readers find these features very appealing.

Unfortunately, the book's informality is arguably taken too far in numerous places, leading to significant problems in several directions; as noted in several online reviews of the book, these include assumptions about prerequisites, clarity, wordiness, thoroughness and some sketchy motivations that are difficult for many readers to grasp (these points are raised in some online reviews of the book, and in my opinion these criticisms are legitimate and constructive; of course, it is also necessary to give appropriate weight to the many positive comments about the book and to remember that, despite the drawbacks, it was chosen as the text for this course). Regarding the overall organization, the numbers of sections in both Chapters 2 and 3 are misleadingly small — each section tends to contain three to six significant topics which arguably deserve to be separate units on their own — and perhaps the supplementary topics could have been integrated into the basic structure of the text more systematically; other choices may have made the book easier to read and understand, but it not at all certain that any alternatives would not have given rise to new problems. In any case, one goal of the course and these notes is to deal with some of the issues mentioned in this paragraph.

Selected additional references. Here are four other references; many others could have been listed, but one has to draw the line somewhere. The first is a book that has been used as a text at UCR and other places in the past, the second is a fairly detailed history of the subject during its formative years from the early 1890s to the early 1950s, and the last two are classic (but not outdated) books; the first book also has detailed historical notes.

J. W. Vick. *Homology Theory*. (Second Edition). *Springer-Verlag, New York etc.*, 1994. ISBN: 3–540–94126–6.

J. Dieudonné. *A History of Algebraic and Differential Topology* (1900 – 1960). *Birkhäuser Verlag, Zurich etc.*, 1989. ISBN: 0–817–63388–X.

S. Eilenberg and N. Steenrod. *Foundations of Algebraic Topology*. (Second Edition). *Princeton University Press, Princeton NJ*, 1952. ISBN: 0–691–07965–X.

E. H. Spanier. *Algebraic Topology*, *Springer-Verlag, New York etc.*, 1994.

The amazon.com sites for Hatcher's and Spanier's books also give numerous other texts in algebraic topology that may be useful.

Finally, there are two other books by Munkres that we shall quote repeatedly throughout these notes. The first will be denoted by [MunkresEDT] and the second by [MunkresAT]; if we simply refer to "Munkres," it will be understood that we mean the previously cited book, *Topology* (Second Edition).

J. R. Munkres. *Elementary differential topology*. (Lectures given at Massachusetts Institute of Technology, Fall, 1961. Revised edition. Annals of Mathematics Studies, No. 54.) *Princeton University Press, Princeton, NJ*, 1966. ISBN: 0–691–09093–9.

J. R. Munkres. *Elements of Algebraic Topology*. *Addison-Wesley, Reading, MA*, 1984. (Reprinted by Westview Press, Boulder, CO, 1993.) ISBN: 0–201–62728–0.

Overview of the course

The course directory file `outline2012.pdf` lists the main topics in the course with references to Hatcher when such references exist. As noted above, the course will begin by building upon the coverage of simplicial complexes and related structures in 205B; this is basically limited to definitions and results that will be needed later in the course. These properties will then be used in the construction of singular homology theory and the proof that it satisfies the axioms presented in 205B; we shall also prove uniqueness results for systems satisfying the axioms and describe additional applications of the theory beyond those of 205B.

At first glance, the next step in the course may seem like formalism gone crazy. Although homology is initially defined to take values in the category of abelian groups, which can be viewed as modules over the integers $\mathbb{Z}$, one can easily modify the definitions to obtain homology theories with coefficients in some field $\mathbb{F}$, which take values in the category of vector spaces over $\mathbb{F}$. For such theories, one can define *cohomology groups* $H^q(X, A; \mathbb{F})$ to be the dual vector spaces to the corresponding homology groups $H_q(X, A; \mathbb{F})$. Since the dual space construction is a contravariant functor, this definition extends to a contravariant functor on pairs of spaces and continuous mappings of pairs.

Why in the world might one want to do this? The following analogies may provide some insight:

(1) When one studies smooth manifolds, the spaces of tangent vectors to points of a manifold are of course central to the subject, but there are also many situations in which it is preferable to work with the dual spaces of *cotangent vectors* or *covectors* at points of the manifold. One key reason for this is that smooth fields of covectors — usually called *differential* 1*-forms* — have many useful formal properties which are at best very awkward to describe in terms of tangent vector fields. Similarly, if we define homology groups with coefficients in a field then their dual spaces turn out to have some nice formal properties which the spaces themselves do not.

(2) A loosely related analogy involves spaces of continuous real valued functions. Given two spaces X and Y with a continuous mapping $f : X \to Y$, the spaces of bounded continuous real valued functions $\mathbf{BC}(X)$ and $\mathbf{BC}(Y)$ can be made into a contravariant functor if we define $f^* : \mathbf{BC}(Y) \to \mathbf{BC}(X)$ so that $f^*(h) = h \circ f$, but usually there is no useful way to make the function spaces into a covariant functor.

It turns out that there is also an extra structure on cohomology groups which has no comparably simple counterpart in homology; namely, we have a functorial multiplicative structure on cohomology groups which is called the **cup product**. As noted on page 185 of Hatcher, these products "are considerably more subtle than the additive structure of cohomology." After defining these products and giving examples which show that they can be highly nontrivial, we shall also give a few applications to homotopy-theoretic questions; we have chosen some applications whose conclusions can be easily stated using concepts from 205A and 205C without mentioning homology or cohomology groups (or fundamental groups).

The last two units deal with the homological and cohomological properties of topological and smooth manifolds. It is unlikely that both can be covered completely in the present course, but each unit deals with fundamentally important results. Unit V proves **de Rham's Theorem**, which states that the cohomology of a smooth manifold can be computed using differential forms. Among other things, this theorem provides a comprehensive setting for answering certain sorts of results which are often stated without proof in multivariable calculus courses like the following:

Theorem. *Let $A \subset \mathbb{R}^3$ be finite, let U be the complement of A, and let $\mathbf{F}$ be a smooth vector field defined on U. Then $\mathbf{F} = \nabla g$ for some smooth function g if and only if its curl satisfies $\nabla \times \mathbf{F} = \mathbf{0}$ (in other words, $\mathbf{F}$ has a potential function if and only if it is irrotational).*

Note that the conclusion fails if, say, we take A to be the z-axis and let $U = \mathbb{R}^3 - A$. In this case the familiar vector field

$$\frac{x\mathbf{j} \ - \ y\mathbf{i}}{x^2 + y^2}$$

is irrotational but is not the gradient of a smooth function defined over all of U (line integrals of this vector field over closed paths are dependent upon the choice of path; if the vector field were a gradient the line integrals would be independent of the choice of path).

Finally, if time permits there will be a Unit VI, which will cover a class of results known as *duality theorems.* One example of such a result is the following:

Simply connected Poincaré duality theorem. *If M^n is a compact simply connected n-manifold and $0 \leq k \leq n$, then the groups $H^k(M^n;\mathbb{F})$ and $H^{n-k}(M^n;\mathbb{F})$ are isomorphic for every field $\mathbb{F}$.*

Note. It is not difficult to check that this result holds in many special cases like products of spheres.

Results like this suggest that homology and cohomology can be applied effectively to study geometrical and topological questions involving manifolds.

Footnote conventions

At a some points of these notes, certain assertions are made without detailed proofs because the details of verifying them are fairly straightforward. In many cases the details are written out in separate files `footnotes`n`.pdf`, where n refers to the unit in question, and a supserscript $^{(\star)}$ denotes a reference to the appropriate file for these details.

I. Further Properties of Simplicial Complexes

Most homology theories for topological spaces can be described using some method of approximating a space X by maps from compact polyhedra into X or maps from X into compact polyhedra. In order to develop such theories, it is necessary to know more about polyhedra and simplicial complexes than we presented in 205B, and accordingly the first unit is devoted to establishing various additional and important facts about simplicial complexes and their (simplicial) homology groups. The first section describes a way of constructing simplicial chains homology that does not require some auxiliary linear ordering of the vertices, and the second shows that every polyhedron in $\mathbb{R}^n$ admits a simplicial decomposition for which the diameters of the simplices are arbitrarily small. In the third section we consider an extremely useful generalization of simplicial complexes called a **finite cell complex** or a **finite CW-complex**, and in Section 4 we prove a fundamentally important result about such complexes known as the *homotopy extension property*, which states that if X is a finite cell complex and $A \subset X$ is a suitably defined subcomplex, then a continuous map f from A to some space Y extends to X if and only if there is a mapping $g : A \to Y$ such that g is homotopic to f and g extends. Finally, in Section 5 we summarize the basic facts about chain homotopies of chain complexes; these objects were defined and studied in the exercises for 205B, but their role in this course is so important that we are restating the main points here.

I.0 : Review

(Hatcher, various sections)

This is a summary of results from Units IV.2–3 from `algtopnotes2012.tex`. At the end of the first part of that course it was clear that algebraic techniques worked very well for spaces called *graphs*. The effectiveness with which such spaces can be studied can be viewed as an example of the following principle:

> Although topological spaces exist in great variety and can exhibit strikingly original properties, the main concern of topology has generally been the study of spaces which are relatively well-behaved.
>
> RS, *Some recent results on topological manifolds*, Amer. Math. Monthly **78** (1971), 941–952.

One goal of `algtopnotes2012.tex` was to define higher dimensional analogs of graphs which can also be studied effectively using algebraic techniques. It turns out that the appropriate generalization involves spaces which, up to homeomorphism, can be built from a class of building blocks called q-dimensional simplices (*sing.* = simplex), where q runs through all nonnegative integers. Spaces which have geometric decompositions of this form were called *polyhedra*, the building blocks were called a *simplicial decomposition*, and the pair of space with decomposition was called a (finite) *simplicial complex*.

The general versions of several key results from vector analysis — namely, Green's Theorem, Stokes' Theorem and the Divergence Theorem — rely heavily on the fact that certain subsets of $\mathbb{R}^2$

and $\mathbb{R}^3$ are nicely homeomorphic to polyhedra; for Green's Theorem, the subsets are regions in the plane with piecewise smooth boundaries, for Stokes' Theorem, the subsets are oriented piecewise smooth surfaces bounded by piecewise smooth curves, and for the Divergence Theorem, the subsets are regions in space whose boundaries are piecewise smooth surfaces (which have outward pointing orientations). It turns out that many important types of topological spaces are homeomorphic to polyhedra; disks and spheres were particularly important examples in 205B. One large and important class of examples is given by the smooth manifolds which are defined and studied in 205C. A proof of this result is given in the second half of [MunkresEDT]. Furthermore, although it is far beyond the scope of the present course to do so, one can also prove that every closed bounded subsets of some $\mathbb{R}^n$ which is *real semialgebraic set* — namely, definable by finitely many real polynomial equations and inequalities — is homeomorphic to a polyhedron. These results combine to show that the class of spaces homeomorphic to polyhedra is broad enough to include many spaces of interest in topology, other branches of mathematics, and even other branches of the sciences. Here is an online reference for the proof of the result on semialgebraic sets and additional background information:

`http://perso.univ-rennes1.fr/michel.coste/polyens/SAG.pdf`

If a space X is homeomorphic to a polyhedron we often say that a *triangulation* of the space consists of a simplicial complex $(P, \mathbf{K})$ and a homeomorphism from P to X.

In Section IV.1 of `algtopnotes2012.tex` we saw that we could recover the isomorphism type of a connected graphs's fundamental group from a purely algebraic construction given by *chain groups*, which are defined in terms of the edges and vertices of the graph. There are analogous algebraic chain groups for simplicial complexes, and one construction for them was given in 205B. There are several motivations for the algebraic definition of boundary homomorphisms which send chains of a given dimension into their boundaries in lower dimensions. For example, in the previously mentioned results from vector analysis the algebraic boundary behaves as follows:

> In Green's Theorem, the boundary takes a suitably oriented sum of all the 2-simplices in the decomposition into a suitably oriented sum of the 1-simplices in the corresponding decomposition of the boundary.
>
> In Stokes's Theorem, the boundary takes a suitably oriented sum of all the 2-simplices in the decomposition into a suitably oriented sum of the 1-simplices in the corresponding decomposition of the boundary.
>
> In the Divergence Theorem, the boundary takes a suitably oriented sum of all the 3-simplices in the decomposition into a suitably oriented sum of the 2-simplices in the corresponding decomposition of the boundary.

In each of the preceding types of examples, it turns out that the algebraic boundaries of the boundary chains are always zero. More generally, this is always the case for the algebraic chains that were defined in 205B for a simplicial complex with respect to a fixed linear ordering of its (finitely many) vertices. Motivated by the 1-dimensional case, one defines a *cycle* to be a chain whose boundary is zero. Since the boundary of a boundary is zero, every boundary chain is automatically a cycle, and one defines **homology groups** to be the quotients of the subgroups of cycles modulo the subgroups of boundaries.

One obvious question with this definition is the reason(s) for wanting to set boundaries equal to zero. Once again vector analysis provides some insight; in some sense the following discussion is not mathematically rigorous because we have not developed all the tools needed to make it complete, but if one does so then all the assertions can be justified. Suppose we have a connected open subset $U \subset \mathbb{R}^3$, and let $\mathbf{F}$ be a smooth vector field defined on U such that its divergence $\nabla \cdot \mathbf{F}$ is zero; this can be viewed as a model for a moving fluid in U which is *incompressible* — the volume around a point neither increases or decreases with the motion — but we do not need this interpretation. Suppose now that we are given two closed surfaces Σ_i in U for $i = 0$ or 1, oriented with suitably defined outward pointing normals. Then we can form the surface integrals of $\mathbf{F} \cdot d\,\Sigma_i$ over the surfaces Σ_i (we shall call these the flux integrals below). Experience suggests that there is a bounded region between these two surfaces if they are disjoint, and in fact one can prove this is always the case. Suppose now that this region is entirely contained in U, so that we can view $\Sigma_0 \cup \Sigma_1$ as the boundary of something in U; if we do this, then for the inner surface the outward pointing normal for the region is the opposite of the usual orientation (think about two concentric spheres). Under these conditions the Divergence Theorem and $\nabla \cdot \mathbf{F} = 0$ imply that the flux integrals of $\mathbf{F}$ over Σ_0 and Σ_1 are the same, for their difference bounds some subregion E of U, and by the divergense theorem the difference of flux integrals is the integral of $\nabla \cdot \mathbf{F} = 0$ over E. So we have the principle that *the flux integrals of two surfaces agree if their difference bounds a region in U.*

The basic identity $d \circ d = 0$ in a simplicial chain complex arises in several contexts, and it is useful to formulate this abstractly as the definition of a *chain complex*. Homology groups given by $H_k :=$ Kernel d_k/Image d_{k+1} can be defined in this generality, and one can prove many useful formal properties. For example, if one defines morphisms of chain complexes in the obvious fashion, then a morphism of chain complexes induces a morphism of homology, and this construction is functorial.

The usefulness of simplicial chain complexes depends upon our ability to compute their homology groups, so the next step is to develop tools for doing so. The boundary homomorphisms in a simplicial chain complex are defined fairly explicitly, and it is not particularly difficult to write a computer program for carrying out the algebraic computations needed to describe simplicial homology groups up to algebraic isomorphism. However, these calculations do not necessarily provide much geometrical insight into the topological structure of a polyhedron, so one also needs further methods which shed more light on such matters.

For example, given a simplicial complex $(P, \mathbf{K})$ and a homology class $u \in H_r(P, \mathbf{K}^\omega)$, one often wants to know if this class is the image of a homology class $u' \in H_r(Q, \mathbf{L}^\omega)$ of some subcomplex $(Q, \mathbf{L}) \subset (P, \mathbf{K})$. For example, if P is a polyhedral region in $\mathbb{R}^3$ and $r = 2$, then one might want to find a 2-dimensional subcomplex with this property; such subcomplexes always exist, but it is often useful to have more specific information. Questions of this sort can often be answered very effectively using *exact sequences of homology groups*. Two types of such sequences were described in 205B, one of which is the *long exact sequence of a pair* consisting of a complex and a subcomplex, and the other of which is the *Mayer-Vietoris exact sequence* which relates the homology of a union of two subcomplexes

$$(P, \mathbf{K}) \;\; = \;\; (P_1, \mathbf{K}_1) \; \cup \; (P_2, \mathbf{K}_2)$$

to the homology of the subcomplexes $(P_i, \mathbf{K}_i)$ and the homology of the intersection subcomplex $(P_1, \mathbf{K}_1) \;\cap\; (P_2, \mathbf{K}_2)$ in much the same way that the Seifert-van Kampen Theorem relates the fundamental group of a union of two open subsets $X = U_1 \cup U_2$ to the fundamental groups of the subspaces U_i and the intersection $U_1 \cap U_2$ provided that all spaces are arcwise connected.

The material discussed thus far can be used very effectively to analyze homology groups of simplicial complexes. However, there is one fundamental point which was not established in 205B:

> TOPOLOGICAL INVARIANCE QUESTION. If P and P' are homeomorphic polyhedra with corresponding simplicial decompositions, are the associated simplicial homology groups isomorphic?

This turns out to be true for graphs because the homology groups are determined by the fundamental groups of the components of the graph, and these fundamental groups of components are isomorphic if the underlying spaces are homeomorphic. For complexes of higher dimension, the problem was avoided by postulating the existence of some construction for homology groups (which we called a *singular homology theory*) which satisfies the topological invariance condition and also has many other important and useful properties. We made this choice for two reasons:

(*i*) The construction requires a substantial amount of time and effort, and the motivation for many of the steps involves properties of simplicial complexes beyond those introduced in 205B. Historically, it took about 50 years for mathematicians to perfect the now definitive approach to constructing the singular homology groups in Hatcher's book (Poincaré's first papers on the subject appeared in the 1890s, and the Eilenberg-Steenrod approach was completed in the 1940s).

(*ii*) One of the strongest motivations for such a construction is an understanding of its usefulness, and the last part of 205B was devoted to using homology groups to prove a few topological results — for example, the fact that open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic if $m \neq n$, the Jordan Curve Theorem which states that a simple closed curve in S^2 separates its complement into two connected components, the Brouwer Fixed Point Theorem, and the fact that certain graphs are not homeomorphic to subsets of $\mathbb{R}^2$. It is often easier to work slowly through some complicated mathematical constructions if their ultimate benefits are understood.

As noted at the beginning of this unit, the first step in constructing a singular homology theory satisfying the axioms in 205B is to formulate and prove results about simplicial complexes that are needed in the construction or are useful in some other respect, and the present unit is devoted to this process. The construction of singular homology will be given in the next unit.

I.1 : Ordered simplicial chains

(Hatcher, § 2.1)

We have already mentioned the topological invariance question, and in fact there is another issue along these lines which is even more basic. The definition of simplicial chains in 205B required the choice of a linear ordering for the vertices, so the first step is to prove that different orderings yield isomorphic homology groups. In order to show this, we have to go back and give alternate definitions of simplicial homology groups which by construction do not involve any choices of vertex orderings. As noted in the 205B notes, this need to redo fundamental definitions frequently is typical of the subject, and it sometimes makes algebraic topology seem like a real-life parody of the film *Groundhog Day* (see `http://www.imdb.com/title/t0107048`).

> Well, it's Groundhog Day ... **again**. ... I was in the Virgin Islands once ... **That** was a pretty good day. Why couldn't I get **that** day over and over and over?

Phil Connors, in the film *Groundhog Day*

Definition. Suppose that $(P, \mathbf{K})$ is a simplicial complex The *unordered simplicial chain group* $C_k(P, \mathbf{K})$ is the free abelian group on all symbols $\mathbf{u}_0 \ \cdots \ \mathbf{u}_k$, where the $\mathbf{u}_j$ are all vertices of some simplex in $\mathbf{K}$ and repetitions of vertices are allowed. A family of differential or boundary homomorphisms d_k is defined as before, and the k-dimensional simplicial homology $H_k(P, \mathbf{K})$ is defined to be the k-dimensional homology of this chain complex.

If ω is a linear ordering for the vertices of $\mathbf{K}$, then the unordered simplicial chain complex $C_*(P, \mathbf{K})$ contains the ordered simplicial chain complex $C_*(P, \mathbf{K}^\omega)$ as a chain subcomplex, and we shall let i denote the resulting inclusion map of chain complexes. If we can show that the associated homology maps i_* are isomorphisms, then it will follow that the homology groups for the ordered simplicial chain complex agree with the corresponding groups for the unordered simplicial chain complex, and therefore the homology groups do not depend upon choosing a linear ordering of the vertices.

One major difference between the unordered and ordered simplicial chain groups is that the latter are nontrivial in every positive dimension. In particular, if $\mathbf{v}$ is a vertex of $\mathbf{K}$, then the free generator $\mathbf{v} \ \cdots \ \mathbf{v} = \mathbf{u}_0 \ \cdots \ \mathbf{u}_k$, with $\mathbf{u}_j = \mathbf{v}$ for all j, represents a nonzero element of $C_k(P, \mathbf{K})$. On the other hand, the ordered simplicial chain groups are nonzero for only finitely many values of k.

In order to analyze the mappings i_*, we shall introduce yet another definition of homology groups.

Third Definition. In the setting above, define the subgroup $C'_k(P, \mathbf{K})$ of *degenerate* simplicial k-chains to be the subgroup generated by

(a) all elements $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k$ such that $\mathbf{v}_i = \mathbf{v}_{i+1}$ for some (at least one) i,

(b) all sums $\mathbf{v}_0 \ \cdots \ \mathbf{v}_i\mathbf{v}_{i+1} \ \cdots \ \mathbf{v}_k \ + \ \mathbf{v}_0 \ \cdots \ \mathbf{v}_{i+1}\mathbf{v}_i \ \cdots \ \mathbf{v}_k$, where $0 \leq i < k$.

We claim these subgroups define a chain subcomplex, and to show this we need to verify the following.

LEMMA 1. *The boundary homomorphism d_k sends elements of $C'_k(P, \mathbf{K})$ to $C'_{k-1}(P, \mathbf{K})$.*

It suffices to prove that the boundary map sends the previously described generators into degenerate chains, and checking this is essentially a routine calculation.■

We now define the complex of *alternating simplicial chains* $C^{\mathrm{alt}}_*(P, \mathbf{K})$ to be the quotient complex $C_*(P, \mathbf{K})/C'_*(P, \mathbf{K})$ with the associated differential or boundary map.

PROPOSITION 2. *The composite $\varphi : C_*(P, \mathbf{K}^\omega) \to C_*(P, \mathbf{K}) \to C^{\mathrm{alt}}_*(P, \mathbf{K})$ is an isomorphism of chain complexes.*

COROLLARY 3. *The morphism $i_* : H_*(P, \mathbf{K}^\omega) \to H_*(P, \mathbf{K})$ is injection onto a direct summand.*

Proof that Proposition 2 implies Corollary 3. Let q be the projection map from unordered to alternating chains, so that $\varphi_* = q_* \circ i_*$. General considerations imply that φ_* is an isomorphism.

Suppose now that $i_*(a) = i_*(b)$. Applying q_* to each side we obtain

$$\varphi_*(a) \quad = \quad q_* \circ i_*(a) \quad = \quad q_* \circ i_*(b) \quad = \quad \varphi_*(b)$$

and since φ_* is bijective it follows that $a = b$.

Now let B_* be the kernel of q_*. We shall prove that every element of $H_*(P, \mathbf{K})$ has a unique expression as $i_*(a) + c$, where $c \in B_*$. Given $u \in H_*(P, \mathbf{K})$, direct computation implies that

$$u \;-\; i_*(\varphi_*)^{-1} q_*(u) \quad \in \quad B_*$$

and thus yields existence. Suppose now that $u = i_*(a) + c$, where $c \in B_*$. It then follows from the definitions that

$$i_*(a) \quad = \quad i_*(\varphi_*)^{-1} q_*(u)$$

and hence we also have

$$c \quad = \quad u \;-\; i_*(a) \quad = \quad u \;-\; i_*(\varphi_*)^{-1} q_*(u)$$

which proves uniqueness.■

Proof of Proposition 2. Analogs of standard arguments for determinants yield the following observations:

(1) *The generator* $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k \in C_k(P, \mathbf{K})$ *lies in the subgroup of degenerate chains if two vertices are equal.*

(2) *If* σ *is a permutation of* $\{0, \ \cdots \ , k\}$, *then* $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k - (-1)^{\mathrm{sgn}(\sigma)} \mathbf{v}_{\sigma(0)} \ \cdots \ \mathbf{v}_{\sigma(k)}$ *is a degenerate chain.*

Define a map of graded abelian groups Ψ from $C_*(P, \mathbf{K})$ to $C_*(P, \mathbf{K}^\omega)$ which sends $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k$ to zero if there are repeated vertices and sends $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k$ to $(-1)^{\mathrm{sgn}(\sigma)} \mathbf{v}_{\sigma(0)} \ \cdots \ \mathbf{v}_{\sigma(k)}$ if the vertices are distinct and σ is the unique permutation which puts the vertices in the proper order:

$$\mathbf{v}_{\sigma(0)} \quad < \quad \cdots \quad < \quad \mathbf{v}_{\sigma(k)}$$

It follows that Ψ passes to a map ψ of quotients from $C_*^{\mathrm{alt}}(P, \mathbf{K})$ to $C_*(P, \mathbf{K}^\omega)$ such that $\psi \circ \varphi$ is the identity. In particular, it follows that φ is injective. To prove it is surjective, note that (1) and (2) imply that $C_k^{\mathrm{alt}}(P, \mathbf{K})$ is generated by the image of φ and hence φ is also surjective. It follows that φ determines an isomorphism of chain complexes as required.■

Acyclic complexes

Definition. An *augmented chain complex* over a ring R consists of a chain complex (C_*, d) and a homomorphism $\varepsilon : C_0 \to R$ (the augmentation map) such that ε is onto and $\varepsilon \circ d_1 = 0$.

All of the simplicial chain complexes defined above have canonical augmentations given by sending expressions of the form $\sum n_{\mathbf{v}} \, \mathbf{v}$ to the corresponding integers $\sum n_{\mathbf{v}}$.

Definition. A simplicial complex is said to be *acyclic* ("has no nontrivial cycles") if $H_j(P, \mathbf{K}) = 0$ for $j \neq 0$ and $H_0 \cong \mathbb{Z}$, with the generator in homology represented by an arbitrary free generator of $C_0(P, \mathbf{K})$.

There is a simple geometric criterion for a simplicial chain complexe to be acyclic.

Definition. A simplicial complex $(P, \mathbf{K})$ is said to be *star shaped* with respect to some vertex $\mathbf{v}$ in $\mathbf{K}$ if for each simplex A in $\mathbf{K}$ either $\mathbf{v}$ is a vertex of A or else there is a simplex $\mathbf{B}$ in $\mathbf{K}$ such that $\mathbf{A}$ is a face of $\mathbf{B}$ and $\mathbf{v}$ is a vertex of $\mathbf{B}$.

Some examples of star shaped complexes are described in `advnotesfigures.pdf` (see Figures ???). One particularly important example for the time being is the standard simplex Δ_n with its standard decomposition.

PROPOSITION 4. *If the simplicial complex $(P, \mathbf{K})$ is star shaped with respect to some vertex, then it is acyclic, and the map $i_* : H_*(P, \mathbf{K}^\omega) \to H_*(P, \mathbf{K})$ is an isomorphism.*

Proof. Define a map of graded abelian groups $\eta : C_*(P, \mathbf{K}) \to C_*(P, \mathbf{K})$ such that $\eta_q : C_q(P, \mathbf{K}) \to C_q(P, \mathbf{K})$ is zero if $q \neq 0$ and η_0 sends a chain y to $\varepsilon(y)\,\mathbf{v}$. Then η is a chain map because $\varepsilon \circ d_1 = 0$.

We next define homomorphisms $D_q : C_q(P, \mathbf{K}) \to C_{q+1}(P, \mathbf{K})$ such that

$$d_{q+1} \circ D_q \quad = \quad \text{identity} \;-\; d_q \circ D_{q-1}$$

if q is positive and

$$d_1 \circ D_0 \quad = \quad \text{identity} \;-\; \eta_0$$

on C_0. We do this by setting $D_q(\mathbf{x}_0 \ \cdots \ \mathbf{x}_q) = \mathbf{v}\mathbf{x}_0 \ \cdots \ \mathbf{x}_q$ and taking the unique extension which exists since the classes $\mathbf{x}_0 \ \cdots \ \mathbf{x}_q$ are free generators for C_q. Elementary calculations show that the mappings D_q satisfy the conditions given above.

To see that $H_q(P, \mathbf{K}) = 0$ if $q > 0$, suppose that $d_q(z) = 0$. Then the first formula implies that $z = d_{q+1} \circ D_q(z)$. Therefore $H_q = 0$ if $q > 0$. On the other hand, if $z \in C_0$, then the second formula implies that $d_1 \circ D_0(z) = z - \varepsilon(z)\,\mathbf{v}$. Furthermore, since $\varepsilon \circ d_1 = 0$ and $d_0 = 0$, it follows that

- (i) the map ε passes to a homomorphism from H_0 to $\mathbb{Z}$,
- (ii) since $\varepsilon(\mathbf{v}) = 1$ this homomorphism is onto,
- (iii) the multiples of the class $[\mathbf{v}]$ give all the classes in H_0.

Taken together, these imply that $H_0(P, \mathbf{K}) \cong \mathbb{Z}$, and it is generated by $[\mathbf{v}]$. This completes the computation of $H_*(P, \mathbf{K})$.

By Corollary 3 we know that $H_q(P, \mathbf{K}^\omega)$ is isomorphic to a direct summand of $H_q(P, \mathbf{K})$ and since the latter is zero if $q > 0$ it follows that the former is also zero if $q > 0$. Similarly, we know that $H_0(P, \mathbf{K}^\omega)$ is isomorphic to a direct summand of $H_0(P, \mathbf{K}) \cong \mathbb{Z}$. By construction we know that the generating class $[\mathbf{v}]$ for the latter lies in the image of i_*, and therefore it follows that the map from $H_0(P, \mathbf{K}^\omega)$ to $H_0(P, \mathbf{K})$ must also be an isomorphism.■

COROLLARY 5. *If Δ is a simplex with the standard simplicial decomposition, then*

$$H_q(\Delta, \mathbf{K}^\omega) \quad \cong \quad H_q(\Delta, \mathbf{K})$$

is trivial if $q \neq 0$ and infinite cyclic if $q = 0$.■

Clearly we would like to "leverage" this result into a proof for an arbitrary simplicial complex $(P, \mathbf{K})$. This will require some additional algebraic tools.

Extension to pairs

Let $\big(\ (P, \mathbf{K}),\ (Q, \mathbf{L})\ \big)$ be a simplicial complex pair consisting of a simplicial complex $(P, \mathbf{K})$ and a subcomplex $(Q, \mathbf{L})$. To simplify notation, we shall often denote such a pair by $(\mathbf{K}, \mathbf{L})$. The unordered simplicial chain complex $C_*(\mathbf{K}, \mathbf{L})$ is defined to be the quotient chain complex $C_*(\mathbf{K})/C_*(\mathbf{L})$, and the unordered relative simplicial homology groups, denoted by $H_*(\mathbf{K}, \mathbf{L})$, are the homlogy groups of these chain complexes. As in the absolute case, we have canonical homomorphisms from the relative homology groups for ordered chains to the relative homology groups for unordered chains. We should also note that the previously defined absolute chain groups may be viewed as special cases of this definition where $\mathbf{L} = \emptyset$.

By the preceding discussion and Theorem V.3.2 from `algtopnotes2012.tex`; (*i.e.*, short exact sequences of chain complexes determine long exact sequences of homology groups), we have the following result:

THEOREM 6. (Long Exact Homology Sequence Theorem — Simplicial Version). *Let* $i : \mathbf{L} \to \mathbf{K}$ *denote a simplicial subcomplex inclusion, and let* ω *be a linear ordering of the vertices. Then there are long exact sequences of homology groups, and they fit into the following commutative diagram, in which the rows are exact and the horizontal arrows represent the canonical maps from ordered to unordered chains:*

$$\begin{array}{ccccccccccc} \cdots & H_{k+1}(\mathbf{K}^\omega, \mathbf{L}^\omega) & \xrightarrow{\partial} & H_k(\mathbf{L}^\omega) & \xrightarrow{i_*} & H_k(\mathbf{K}^\omega) & \xrightarrow{j_*} & H_k(\mathbf{K}^\omega, \mathbf{L}^\omega) & \xrightarrow{\partial} & H_{k-1}(\mathbf{L}^\omega) & \cdots \\ & \downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow \varphi_* & \\ \cdots & H_{k+1}(\mathbf{K}, \mathbf{L}) & \xrightarrow{\partial} & H_k(\mathbf{L}) & \xrightarrow{i_*} & H_k(\mathbf{K}) & \xrightarrow{j_*} & H_k(\mathbf{K}, \mathbf{L}) & \xrightarrow{\partial} & H_{k-1}(\mathbf{L}) & \cdots \end{array}$$

Sketch of proof. The definitions of simplicial chain groups imply that one has a commutative diagram of short exact sequences which goes from the ordered chain complex short exact sequence

$$0 \ \to \ C_*(\mathbf{L}^\omega) \ \to \ C_*(\mathbf{K}^\omega) \ \to \ C_*(\mathbf{K}^\omega, \mathbf{L}^\omega) \ \to \ 0$$

to the unordered chain complex short exact sequence

$$0 \ \to \ C_*(\mathbf{L}) \ \to \ C_*(\mathbf{K}) \ \to \ C_*(\mathbf{K}, \mathbf{L}) \ \to \ 0 \ .$$

The theorem follows by taking the associated long exact homology sequences and using the naturality of these sequences with respect to maps of short exact sequences of chain complexes.■

At this point it is also appropriate to recall another result on diagrams with exact sequences from `algtopnotes2012.tex`; namely, the Five Lemma (Theorem V.3.4).

The isomorphism theorem

Here is the result that has been our main objective:

THEOREM 7. *If* $(\mathbf{K}, \mathbf{L})$ *is a simplicial complex pair, then the canonical map*

$$\varphi_* : H_*(\mathbf{K}^\omega, \mathbf{L}^\omega) \ \to \ H_*(\mathbf{K}, \mathbf{L})$$

is an isomorphism.

Proof. Consider the following statements:

> $(\mathbf{X}_n)$ *The map* φ *above is an isomorphism for all simplicial complex pairs* $(\mathbf{K}, \mathbf{L})$ *such that* $\dim \mathbf{K} \leq n$.
>
> $(\mathbf{Y}_{n+1})$ *The map* φ *above is an isomorphism for all* $(\mathbf{K}, \mathbf{L})$ *such that* $\dim \mathbf{K} \leq n$ *and also for* $(\Delta_{n+1}, \partial\Delta_{n+1})$.
>
> $(\mathbf{W}_{n+1,m})$ *The map* φ *above is an isomorphism for all* $(\mathbf{K}, \mathbf{L})$ *such that* $\dim \mathbf{K} \leq n$ *and also for all* $(\mathbf{K}, \mathbf{L})$ *such that* $\dim \mathbf{K} \leq n+1$ *and* $\mathbf{K}$ *has at most* m *simplices of dimension equal to* $n+1$.

The theorem is then established by the following double inductive argument:

[F] The statement $(\mathbf{X}_0)$ and the equivalent statement $(\mathbf{W}_{1,0})$ are true.

[G] For all nonnegative integers n, the statement $(\mathbf{X}_n)$ implies $(\mathbf{Y}_{n+1})$.

[K] For all nonnegative integers n and m, the statements $(\mathbf{W}_{n+1,m})$ and $(\mathbf{Y}_{n+1})$ imply $(\mathbf{W}_{n+1,m+1})$.

Since statement $(\mathbf{X}_n)$ is true if and only if $(\mathbf{W}_{n,m})$ is true for all m, and the latter are all true if and only if $(\mathbf{W}_{n+1,0})$ is true, we also have the following:

[L] For all n the statements $(\mathbf{X}_n) \Longleftrightarrow (\mathbf{W}_{n+1,0})$ and $(\mathbf{Y}_{n+1})$ imply $(\mathbf{W}_{n+1,m})$ for all m, and hence $(\mathbf{X}_n)$ implies $(\mathbf{X}_{n+1})$.

Therefore $(\mathbf{X}_n)$ is true for all n, and this is the conclusion of the theorem.

Proof of [F]. By the Five Lemma it suffices to prove the result when $\mathbf{L}$ is empty. Since the 0-dimensional complex determined by $\mathbf{K}$ is merely a finite set of vertices, write these vertices as $\mathbf{w}_1, \cdots \mathbf{w}_m$. We then have canonical chain complex isomorphisms

$$\bigoplus_{j=1}^{m} C_*(\{\mathbf{w}_j\}^\omega) \longrightarrow C_*(\mathbf{K}^\omega)\ , \qquad \bigoplus_{j=1}^{m} C_*(\{\mathbf{w}_j\}) \longrightarrow C_*(\mathbf{K})$$

and these pass to homology isomorphisms

$$\bigoplus_{j=1}^{m} H_*(\{\mathbf{w}_j\}^\omega) \longrightarrow H_*(\mathbf{K}^\omega)\ , \qquad \bigoplus_{j=1}^{m} H_*(\{\mathbf{w}_j\}) \longrightarrow H_*(\mathbf{K})\ .$$

These maps commute with the homomorphisms φ_* sending ordered to unordered chains. and since the maps φ_* are isomorphisms for one point complexes (= 0-simplices), it follows that φ defines an isomorphism from $H_*(\mathbf{K}^\omega)$ to $H_*(\mathbf{K})$. The completes the proof of $(\mathbf{X}_0)$.

Proof of [G]. By $(\mathbf{X}_n)$ we know that φ_* is an isomorphism for the complex $\partial\Delta_{n+1}$. Since φ_* is also an isomorphism for Δ_{n+1} by Corollary III.3.6. Therefore the Five Lemma implies that φ_* is an isomorphism for $(\Delta_{n+1}, \partial\Delta_{n+1})$.

Proof of [K]. This is the crucial step. Let $\mathbf{K}$ be an $(n+1)$-dimensional complex, and let $\mathbf{M}$ be a subcomplex obtained by removing exactly one $(n+1)$-simplex from $\mathbf{K}$, so that φ_* is an isomorphism for $\mathbf{M}$ by the inductive hypothesis. If we can show that φ_* is an isomorphism for $(\mathbf{K}, \mathbf{M})$, then it will follow that φ_* is an isomorphism for $\mathbf{K}$, and the relative case will the follow from the Five Lemma.

Let $\mathbf{S}$ be the extra simplex of $\mathbf{K}$ and let $\partial\mathbf{S}$ be its boundary. Then there are canonical isomorphism from the chain groups of Δ_{n+1}, $\partial\Delta_{n+1}$ and $(\Delta_n, \partial\Delta_{n+1})$ to the chain groups of $\mathbf{S}$, $\partial\mathbf{S}$ and $(\mathbf{S}, \partial\mathbf{S})$. We then have the following commutative diagram, in which the morphisms α and β are determined by subcomplex inclusions:

$$\begin{array}{ccc} C_*(\mathbf{S}^\omega, \partial\mathbf{S}^\omega) & \xrightarrow{\ \alpha\ } & C_*(\mathbf{K}^\omega, \mathbf{M}^\omega) \\ \downarrow \varphi(\mathbf{S}, \partial\mathbf{S}) & & \downarrow \varphi(\mathbf{K}, \mathbf{M}) \\ C_*(\mathbf{S}, \partial\mathbf{S}) & \xrightarrow{\ \beta\ } & C_*(\mathbf{K}, \mathbf{M}) \end{array}$$

We CLAIM that α and β are isomorphisms of chain complexes. For the mapping α, this follows because the relative ordered chain groups of a pair $(\mathbf{T}, \mathbf{T}_0)$ are free abelian groups on the simplices in $\mathbf{T} - \mathbf{T}_0$, and each of the sets $\mathbf{S} - \partial\mathbf{S}$ and $\mathbf{K} - \mathbf{M}$ is given by the same $(n+1)$-simplex. For the mapping β, this follows because the relative unordered chain groups of a pair $(\mathbf{T}, \mathbf{T}_0)$ are free abelian groups on the generators $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k$, where the $\mathbf{v}_j$ are vertices of a simplex that is in $\mathbf{T}$ but not in $\mathbf{T}_0$ (with repetitions allowed as usual), and once again these free generators are identical for te pairs $(\mathbf{S}, \partial\mathbf{S})$ and $(\mathbf{K}, \mathbf{M})$ because $\mathbf{S} - \partial\mathbf{S}$ and $\mathbf{K} - \mathbf{M}$ are the same.

By $(\mathbf{Y}_{n+1})$ we know that $\varphi(\mathbf{S}, \partial\mathbf{S})$ defines an isomorphism in homology, and therefore it follows that the homology map

$$\varphi(\mathbf{K}, \mathbf{M})_* \quad = \quad \beta_* \circ \varphi(\mathbf{S}, \partial\mathbf{S})_* \circ \alpha_*^{-1}$$

also defines an isomorphism in homology. We can now use the Five Lemma and $(\mathbf{W}_{n+1,m})$ to conclude that the map $\varphi(\mathbf{K})$ defines an isomorphism in homology, and finally we can use the Five Lemma once more to see that the statement $(\mathbf{W}_{n+1,m+1})$ is true. This completes the proof of [K], and as noted above it also yields [L] and the theorem.■

The preceding result can be reformulated in an abstract setting that will be needed later. We begin by defining a category **SCPairs** whose objects are pairs of simplicial complexes $(\mathbf{K}, \mathbf{K}_0)$ and whose morphisms are given by subcomplex inclusions $(\mathbf{L}, \mathbf{L}_0) \subset (\mathbf{K}, \mathbf{K}_0)$; in other words, $\mathbf{L}_0$ is a subcomplex of both $\mathbf{L}$ and $\mathbf{K}_0$ while $\mathbf{L}$ is also a subcomplex of $\mathbf{K}$. A *homology theory* on this category is a covariant functor h_* valued in some category of modules together with a natural transformation

$$\partial(\mathbf{K}, \mathbf{L}) : h_*(\mathbf{K}, \mathbf{L}) \ \longrightarrow \ h_{*-1}(\mathbf{L})$$

such that

(*a*) one has long exact homology sequences,

(*b*) if $\mathbf{K}$ is a simplex and $\mathbf{v}$ is a vertex of $\mathbf{K}$ then $h_*(\{\mathbf{v}\}) \to h_*(\mathbf{K})$ is an isomorphism,

(*c*) if $\mathbf{K}$ is 0-dimensional with vertices $\mathbf{v}_j$ then the associated map from $\oplus_j \ h_j(\{\mathbf{v}_j\})$ to $h_*(\mathbf{K})$ is an isomorphism,

(*d*) if $\mathbf{K}$ is obtained from $\mathbf{M}$ by adding a single simplex $\mathbf{S}$, then $h_*(\mathbf{S}, \partial\mathbf{S}) \to h_*(\mathbf{M}, \mathbf{K})$ is an isomorphism,

(*d*) if $\mathbf{K}$ is complex consisting only of a single vertex then $h_0(\mathbf{K})$ is the underlying ring R and $h_j(\mathbf{K}) = 0$ if $j \neq 0$.

A *natural transformation* from one such theory (h_*, ∂) to another (h'_*, ∂') is a natural transformation of θ of functors that is compatible with the mappings ∂ and ∂'; specifically, we want

$$\theta(\mathbf{L}) \circ \partial \quad = \quad \partial' \circ \theta(\mathbf{K}, \mathbf{L}) \ .$$

These conditions imply the existence of a commutative ladder diagram as in Theorem 6, where the rows are the long exact sequences determined by the two abstract homology theories. The definition is set up so that the proof of the next result is formally parallel to the proof of Theorem 7:

THEOREM 8. *Suppose we are given a natural transformation of homology theories θ as above such that $\theta(\mathbf{K})$ is an isomorphism if $\mathbf{K}$ consists of just a single vertex. Then $\theta(\mathbf{K}, \mathbf{L})$ is an isomorphism for all pairs $(\mathbf{K}, \mathbf{L})$.*■

I.2 : Subdivisions

(Hatcher, § 2.1)

For many purposes it is convenient or necessary to replace a simplicial decomposition **K** of a polyhedron P by another decomposition **L** with smaller simplices. More precisely, we would like the smaller simplices in **L** to determine simplicial decompositions for each of the simplices in **K**.

The need for working with subdivisions arises in many contexts. For example, as Figure ??? in `advnotesfigures.pdf` suggests, the union of two solid triangular regions in the plane usually does not satify the conditions for a simplicial decomposition, but it is possible to subdivide the union and obtain a simplicial decomposition such that each of the original regions is a subcomplex. Similar considerations apply to arbitrary polyhedra. We shall not attempt to state this precisely or prove it because we do not need such results in this course, but here are some references:

J. F. P. Hudson. *Piecewise Linear Topology*. *W. A. Benjamin, New York*, 1969. (Online: `http://www.maths.ed.ac.uk/∼aar/surgery/hudson.pdf`)

C. P. Rourke and B. J. Sanderson. *Introduction to Piecewise-Linear Topology (Ergeb. Math. Bd. 69). Springer-Verlag, New York–etc.*, 1972.

A few topics are also discussed in [MunkresEDT]. An extremely detailed study of some topics in this section also appears in the following online book:

`http://www.cis.penn.edu/∼jean/gbooks/convexpoly.html`

Explicit simple examples

1. If P is a 1-simplex with vertices $\mathbf{x}$ and $\mathbf{y}$, and **K** is the standard decomposition given by P and the endpoints, then there is a subdivision **L** given by trisecting P; specifically, the vertices are given by $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} = \frac{2}{3}\mathbf{x} + \frac{1}{3}\mathbf{y}$, and $\mathbf{w} = \frac{1}{3}\mathbf{x} + \frac{2}{3}\mathbf{y}$, and the 1-simplices are $\mathbf{xw}$, $\mathbf{wz}$ and $\mathbf{zy}$. This is illustrated as Figure ??? in the file `advnotesfigures.pdf`.

2. Similarly, if $[a, b]$ is a closed interval in the real line and we are given a finite sequence $a = t_0 < \cdots < t_m = b$, then these points and the intervals $[t_{j-1}, t_j]$, where $1 \leq j \leq n$, form a subdivision of the standard decomposition of $[a, b]$.

3. If P is the 2-simplex with vertices $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, and **K** is the standard decomposition given by P and its faces, then there is an obvious decomposition **L** which splits P into two simplices $\mathbf{xyz}$ and $\mathbf{xyw}$, where $\mathbf{w} = \frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z}$ is the midpoint of the 1-simplex $\mathbf{yz}$. Similar eamples exist if we take $\mathbf{z} = a\mathbf{y} + (1 - a)\mathbf{z}$, where a is an arbitrary number such that $0 < a < 1$ (see Figure ??? in the file `advnotesfigures.pdf`).

Formal definition of subdivisions

Each of the preceding examples is consistent with the following general concept.

Definition. Let $(P, \mathbf{K})$ be a simplicial complex, and let **L** be a simplicial decomposition of P. Then **L** is called a (linear) *subdivision* of **K** if every simplex of **L** is contained in a simplex of **K**.

The following observation is very elementary, but we shall need it in the discussion below.

PROPOSITION 0. *Suppose P is a polyhedron with simplicial decompositions $\mathbf{K}$, $\mathbf{L}$ and $\mathbf{M}$ such that $\mathbf{L}$ is a subdivision of $\mathbf{K}$ and $\mathbf{M}$ is a subdivision of $\mathbf{L}$. Then $\mathbf{M}$ is also a subdivision of $\mathbf{K}$.*■

Figure ??? in `advnotesfigures.pdf` depicts two subdivisions of a 2-simplex that are different from the one in Example 3 above. As indicated by Figure ??? in the same document, in general if we have two simplicial decompositions of a polyhedron then neither is a subdivision of the other. However, it is possible to prove the following:

> *If $\mathbf{K}$ and $\mathbf{L}$ are simplicial decompositions of the same polyhedron P, then there is a third decomposition which is a subdivision of both $\mathbf{K}$ and $\mathbf{L}$.*

Proving this requires more machinery than we need for other purposes, and since we shall not need the existence of such subdivisions in this course we shall simply note that one can prove this result using methods from the second part of [MunkresEDT]:

SUBDIVISION AND SUBCOMPLEXES. These two concepts are related by the following elementary results.

PROPOSITION 1. *Suppose that $(P, \mathbf{K})$ is a simplicial complex and that $(P_1, \mathbf{K}_1)$ is a subcomplex of $(P, \mathbf{K})$. If $\mathbf{L}$ is a subdivision of $\mathbf{K}$ and $\mathbf{L}_1$ is the set of all simplices in $\mathbf{L}$ which are contained in P_1, then $(P_1, \mathbf{L}_1)$ is a subcomplex of $(P, \mathbf{L})$.*■

Recall our Default Hypothesis (at the end of Section I.2) that all simplicial decompositions should be closed under taking faces unless specifically stated otherwise.

COROLLARY 2. *Let P, $\mathbf{K}$ and $\mathbf{L}$ be as above, and let $A \subset P$ be a simplex of $\mathbf{K}$. Then $\mathbf{L}$ determines a simplicial decomposition of A.*■

Barycentric subdivisions

We are particularly interested in describing a systematic construction for subdivisions that works for all simplicial complexes and allows one to form decompositions for which the diameters of all the simplices are very small. This will generalize a standard method for partitioning an interval $[a, b]$ into small intervals by first splitting the interval in half at the midpoint, then splitting the two subintervals in half similarly, and so on. If this is done n times, the length of each interval in the subdivision is equal to $(b - a)/2^n$, and if $\varepsilon > 0$ is arbitrary then for sufficiently large values of n the lengths of the subintervales will all be less than ε.

The generalization of this to higher dimensions is called the **barycentric subdivision**.

Definition. Given an n-simplex $A \subset \mathbb{R}^m$ with vertices $\mathbf{v}_0, \cdots, \mathbf{v}_n$, the *barycenter* $\mathbf{b}_A$ of A is given by

$$\mathbf{b}_A \quad = \quad \frac{1}{n+1} \sum_{i=0}^{n} \mathbf{v}_i \ .$$

If $n \le m \le 3$, this corresponds to the physical center of mass for A, assuming the density in A is uniform.

Definition. If $P \subset \mathbb{R}^m$ is a polyhedron and $(P, \mathbf{K})$ is a simplicial complex, then the *barycentric subdivision* $\mathbf{B(K)}$ consists of all simplices having the form $\mathbf{b}_0 \cdots \mathbf{b}_k$, where (i) each $\mathbf{b}_j$ is the barycenter of a simplex $A_j \in \mathbf{K}$, (ii) for each $j > 0$ the simplex A_{j-1} is a face of A_j.

In order to justify this definition, we need to prove the following result:

PROPOSITION 3. *Let A be an n-simplex, suppose that we are given simplices $A_j \subset A$ such that A_{j-1} is a face of A_j for each j, and let $\mathbf{b}_j$ be the barycenter of A_j. Then the set of vertices $\{\mathbf{b}_0, \cdots, \mathbf{b}_q\}$ is affinely independent.*

Proof. We can extend the sequence of simplices $\{A_j\}$ to obtain a new sequence $C_0 \subset \cdots \subset C_n = A$ such that each C_k is obtained from the preceding one C_{k-1} by adding a single vertex, and it suffices to prove the result for the corresponding sequence of barycenters. Therefore we shall assume henceforth in this proof that each A_j is obtained from its predecessor by adding a single vertex and that A is the last simplex in the list.

It suffices to show that the vectors $\mathbf{b}_j - \mathbf{b}_0$ are linearly independent. For each j let $\mathbf{v}_{j_i}$ be the vertex in A_j that is not in its predecessor. Then for each $j > 0$ we have

$$\mathbf{b}_j - \mathbf{b}_0 \quad = \quad \left(\frac{1}{j+1} \sum_{k \leq j} \mathbf{v}_{i_k} \right) \; - \; \mathbf{v}_0 \quad = \quad \frac{1}{j+1} \sum_{k \leq j} (\mathbf{v}_{i_k} - \mathbf{v}_{i_0}) \; .$$

which is a linear combination of the linearly independent vectors $\mathbf{v}_{i_1} - \mathbf{v}_{i_0}, \cdots, \mathbf{v}_{i_j} - \mathbf{v}_{i_0}$ such that the coefficient of the last vector in the set is nonzero.

If we let $\mathbf{u}_k = \mathbf{v}_{i_k} - \mathbf{v}_{i_0}$, then it follows that for all $k > 0$ we have $\mathbf{b}_k - \mathbf{b}_0 = a_k \mathbf{u}_k + \mathbf{y}_k$, where $\mathbf{y}_k$ is a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_{k-1}$ and $a_k \neq 0$. Since the vectors $\mathbf{u}_j$ are linearly independent, it follows that the vectors $\mathbf{b}_k - \mathbf{b}_0$ (where $0 < k \leq n$) are linearly independent and hence the vectors $\mathbf{b}_0, \cdots, \mathbf{b}_n$ are affinely independent.∎

The simplest nontrivial examples of barycentric subdivisions are given by 2-simplices, and Figure ??? in `advnotesfigures.pdf` gives a typical example. We shall enumerate the simplices in such a barycentric subdivision using the definition. For the sake of definiteness, we shall call the simplex P and the vertices $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$.

(*i*) The 0-simplices are merely the barycenters $\mathbf{b}_A$, where A runs through all the nonempty faces of P and P itself. There are 7 such simplices and hence 7 vertices in $\mathbf{B(K)}$.

(*ii*) The 1-simplices have the form $\mathbf{b}_A\mathbf{b}_C$, where A is a face of C. There are three possible choices for the ordered pair $(\dim A, \dim C)$; namely, $(0,1)$, $(0,2)$ and $(1,2)$. The number of pairs $\{A,\ C\}$ for the case $(0,1)$ is equal to 6, the number for the case $(0,2)$ is equal to 3, and the number for the case $(0,1)$ is also equal to 3, so there are 12 different 1-simplices in $\mathbf{B(K)}$.

(*iii*) The 2-simplices have the form $\mathbf{b}_A\mathbf{b}_C\mathbf{b}_E$, where A is a face of C and C is a face of E. There are 6 possible choices for $\{A,\ C,\ E\}$.

Obviously one could carry out a similar analysis for a 3-simplex but the details would be more complicated.

Of course, it is absolutely essential to verify the that barycentric subdivision construction actually defines simplicial decompositions.

THEOREM 4. *If $(P, \mathbf{K})$ is a simplicial complex and $\mathbf{B(K)}$ is the barycentric subdivision of $\mathbf{K}$, then $(P, \mathbf{B(K)})$ is also a simplicial complex (in other words, the collection $\mathbf{B(K)}$ determines a simplicial decomposition of P).*

Several steps in the proof of this result are fairly intricate, and the following remark from Davis and Kirk, *Lecture Notes in Algebraic Topology*, are worth remembering:

By their second year of graduate studies students must make the transition from understanding simple proofs line-by-line to understanding the overall structure of proofs of [long or] difficult theorems. [Of course it is still necessary to understand simple proofs in detail, but as one progresses it is necessary to begin the study of more complicated arguments by having some grasp of the main steps and how they are studied.]

Proof. We shall concentrate on the special case where P is a simplex. The general case can be recovered from the special case and Lemma IV.2.6 in `algtop-notes.pdf` (see p. 51).

Suppose now that P is a simplex with vertices vertices $\mathbf{v}_0, \cdots, \mathbf{v}_n$. We first show that P is the union of the simplices in $\mathbf{B(K)}$. Given $\mathbf{x} \in P$, write $\mathbf{x}$ as a convex combination $\sum_j t_j \mathbf{v_j}$, and rearrange the scalars into a sequence

$$t_{k_0} \geq t_{k_1} \cdots \geq t_{k_n}$$

(this is not necessarily unique, and in particular it is not so if $t_u = t_v$ for $u \neq v$). For each i between 0 and n, let A_i be the simplex whose vertices are $\mathbf{v}_{k_0}, \cdots, \mathbf{v}_{k_i}$. We CLAIM that $x \in \mathbf{b}_0 \cdots \mathbf{b}_n$, where $\mathbf{b}_i$ is the barycenter of A_i.

Let $s_i = t_{k_i} - t_{k_{i+1}}$ for $0 \leq i \leq n-1$ and set $s_n = t_{k_n}$. Then $s_i \geq 0$ for all i, and it is elementary to verify that

$$\mathbf{x} = \sum_{i=0}^{n} (i+1)\, s_i\, \mathbf{b}_i , \qquad \text{where} \qquad \sum_{1=0}^{n} (i+1)\, s_i = \sum_{i=0}^{n} t_{k_i} = 1 .$$

Therefore $\mathbf{x} \in \mathbf{b}_0 \cdots \mathbf{b}_n$, so that every point in A lies on one of the simplices in the barycentric subdivision.

To conclude the proof, we must show that the intersection of two simplices in $\mathbf{B(K)}$ is a common face. First of all, it suffices to show this for a pair of n-dimensional simplices; this follows from the argument following the Default Hypothesis at the end of Section IV.2 in `algtop-notes.pdf`.

Suppose now that α and γ are n-simplices in $\mathbf{B(K)}$. Then the vertices of α are barycenters of simplices $A_0, \cdots, A_n$ where A_j has one more vertex than A_{j-1} for each j, and the vertices of γ are barycenters of simplices $C_0, \cdots, C_n$ where C_j has one more vertex than C_{j-1} for each j. Label the vertices of the original simplex as $\mathbf{v}_{i_0}, \cdots, \mathbf{v}_{i_n}$ where $A_j = \mathbf{v}_{i_0} \cdots \mathbf{v}_{i_j}$ and also as $\mathbf{v}_{k_0}, \cdots, \mathbf{v}_{k_n}$ where $C_j = \mathbf{v}_{k_0} \cdots \mathbf{v}_{k_j}$. The key point is to determine how $(i_0, \cdots, i_n)$ and $(k_0, \cdots, k_n)$ are related.

If $\mathbf{x}$ lies on the original simplex and $\mathbf{x}$ is written as a convex combination $\sum_j t_j \mathbf{v}_j$, then we have shown that $\mathbf{x} \in A$ if $t_{i_0} \leq \cdots \leq t_{i_n}$. In fact, we can reverse the steps in that argument to show that if $\mathbf{x} \in A$ then conversely we have $t_{i_0} \leq \cdots \leq t_{i_n}$. Similarly, if $\mathbf{x} \in C$ then $t_{k_0} \leq \cdots \leq t_{k_n}$. Therefore if $\mathbf{x} \in A \cap C$ then $t_{i_j} = t_{k_j}$ for all j. Choose $m_0, \cdots, m_q \in \{0, \cdots, n\}$ such that $t_{m_j} > t_{m_{j+1}}$, with the convention that $t_{n+1} = 0$, and split $\{0, \cdots, {}_n\}$ into equivalence classes $\mathcal{M}_0, \cdots, \mathcal{M}_q$ such that $\mathcal{M}_j$ is the set of all u such that $t_u = t_{m_j}$. It follows that $\mathbf{x}$ lies on the simplex $\mathbf{z}_0 \cdots \mathbf{z}_q$, where $\mathbf{z}_j$ is the barycenter of the simplex whose vertices are $\mathcal{M}_0 \cup \cdots \cup \mathcal{M}_j$. The vertices of this simplex are vertices of both A and C. Since $A \cap C$ is convex, this implies that it is the simplex whose vertices are those which lie in $A \cap C$, and thus $A \cap C$ is a face of both A and C.■

Terminology. Frequently the complex $(P, \mathbf{B}(\mathbf{K}))$ is called the *derived complex* of $(P, \mathbf{K})$. The barycentric subdivision construction can be iterated, and thus one obtains a sequence of decompositions $\mathbf{B}^r(\mathbf{K})$. The latter is often called the r^{th} barycentric subdivision of $\mathbf{K}$ and $(P, \mathbf{B}^r(\mathbf{K}))$ is often called the r^{th} derived complex of $(P, \mathbf{K})$.

Diameters of barycentric subdivisions

Given a metric space $(X, \mathbf{d})$, its *diameter* is the least upper bound of the distances $\mathbf{d}(y, z)$, where $y, z \in X$; if the set of distances is unbounded, we shall follow standard usage and say that the diameter is infinite or equal to ∞.

PROPOSITION 5. *Let $A \subset \mathbb{R}^n$ be an n-simplex with vertices $\mathbf{v}_0, \cdots, \mathbf{v}_n$. Then the diameter of A is the maximum of the distances $|\mathbf{v}_i - \mathbf{v}_j|$, where $0 \le i, j \le n$.*

Proof. Let $\mathbf{x}, \mathbf{y} \in A$, and write these as convex combinations $\mathbf{x} = \sum_j t_j \mathbf{v}_j$ and $\mathbf{y} = \sum_j s_j \mathbf{v}_j$. Then

$$\mathbf{x} - \mathbf{y} \quad = \quad \left(\sum_i s_i\right) \mathbf{x} \; - \; \left(\sum_j t_j\right) \mathbf{y} \quad = \quad \sum_{i,j} s_i t_j \, \mathbf{v}_j \; - \; \sum_{i,j} s_i t_j \, \mathbf{v}_i \; .$$

Since $0 \le s_i, t_j \le 1$ for all i and j, we have $0 \le s_i t_j \le 1$ for all i and j, so that

$$\mathbf{d}(\mathbf{x}, \mathbf{y}) \quad = \quad |\mathbf{x} - \mathbf{y}| \quad \le \quad \Big| \sum_{i,j} s_i t_j \, (\mathbf{x}_j - \mathbf{x}_i) \Big| \quad \le$$

$$\sum_{i,j} s_i t_j \, |\mathbf{v}_i - \mathbf{v}_j| \quad \le \quad \sum_{i,j} s_i t_j \max |\mathbf{v}_k - \mathbf{v}_\ell| \quad = \quad \max |\mathbf{v}_k - \mathbf{v}_\ell|$$

as required.■

Definition. If $\mathbf{K}$ is a simplicial decomposition of a polyhedron P, then the *mesh* of $\mathbf{K}$, written $\mu(\mathbf{K})$, is the maximum diameter of the simplices in $\mathbf{K}$.

PROPOSITION 6. *In the preceding notation, the mesh of $\mathbf{K}$ is the maximum distance $|\mathbf{v} - \mathbf{w}|$, where $\mathbf{v}$ and $\mathbf{w}$ are vertices of some simplex in $\mathbf{K}$.*■

The main result in this discussion is a comparison of the mesh of $\mathbf{K}$ with the mesh of $\mathbf{B}(\mathbf{K})$.

PROPOSITION 7. *Suppose that $(P, \mathbf{K})$ be a simplicial complex and that all simplices of $\mathbf{K}$ have dimension $\le n$. Then*

$$\mu(\, \mathbf{B}(\mathbf{K})\,) \quad \le \quad \frac{n}{n+1} \cdot \mu(\mathbf{K}) \; .$$

Before proving this result, we shall derive some of its consequences.

COROLLARY 8. *In the preceding notation, if $r \ge 1$ then*

$$\mu(\, \mathbf{B}^r(\mathbf{K})\,) \quad \le \quad \left(\frac{n}{n+1}\right)^r \cdot \mu(\mathbf{K}) \; .■$$

COROLLARY 9. *In the preceding notation, if $\varepsilon > 0$ then there exists an r_0 such that $r \ge r_0$ implies $\mu(\, \mathbf{B}^r(\mathbf{K})\,) \; < \; \varepsilon$.*

Corollary 9 follows from Corollary 8 and the fact that

$$\lim_{r \to \infty} \left(\frac{n}{n+1} \right)^r \quad = \quad 0 \ . \blacksquare$$

Proof of Proposition 7. By Proposition 5 and the definition of barycentric subdivision we know that $\mu(\,\mathbf{B}(\mathbf{K})\,)$ is the maximum of all distances $|\mathbf{b}_A - \mathbf{b}_C|$, where $\mathbf{b}_A$ and $\mathbf{b}_C$ are barycenters of simplices $A,\ C \in \mathbf{K}$ such that $A \subset C$. Suppose that A is an a-simplex and C is a c-simplex, so that $0 \leq a < c \leq n$. We then have

$$|\mathbf{b}_A - \mathbf{b}_C| \quad = \quad \left| \frac{1}{a+1} \sum_{\mathbf{v} \in A} \mathbf{v} \ - \ \frac{1}{c+1} \sum_{\mathbf{w} \in C} \mathbf{w} \right|$$

and as in the proof of Proposition 5 we have

$$\frac{1}{a+1} \sum_{\mathbf{v} \in A} \mathbf{v} \ - \ \frac{1}{c+1} \sum_{\mathbf{w} \in C} \mathbf{w} \quad = \quad \frac{1}{(a+1)(c+1)} \sum_{\mathbf{v},\mathbf{w}} (\mathbf{v} - \mathbf{w}) \ .$$

There are $(a+1)$ terms in this summation which vanish (namely, those for which $\mathbf{w} = \mathbf{v}$), and therefore we have

$$|\mathbf{b}_A - \mathbf{b}_C| \quad = \quad \left| \frac{1}{(a+1)(c+1)} \sum_{\mathbf{v} \neq \mathbf{w}} (\mathbf{v} - \mathbf{w}) \right| \quad \leq \quad \frac{1}{(a+1)(c+1)} \sum_{\mathbf{v} \neq \mathbf{w}} |\mathbf{v} - \mathbf{w}| \quad \leq$$

$$\frac{1}{(a+1)(c+1)} \cdot \left(\max_{\mathbf{v},\mathbf{w}} \right) |\mathbf{v} - \mathbf{w}| \cdot \Big[(a+1)(c+1) - (a+1) \Big] \quad =$$

$$\left(\max_{\mathbf{v},\mathbf{w}} |\mathbf{v} - \mathbf{w}| \right) \cdot \left(1 \ - \ \frac{1}{c+1} \right) \quad \leq \quad \left(1 \ - \ \frac{1}{n+1} \right) \ .$$

At the last step we use $c \leq n$ and the fact that the function $1 - (1/x)$ is an increasing function of x if $x > 1$. The inequality in the corollary follows directly from the precedng chain of inequalities.$\blacksquare$

One further consequence of Proposition 7 will be important for our purposes.

COROLLARY 10. *Let $(P, \mathbf{K})$ be a simplicial complex, and let $\mathcal{W}$ be an open covering of P. Then there is a positive integer r_0 such that $r \geq r_0$ implies that every simplex of $\mu(\,\mathbf{B}^r(\mathbf{K})\,)$ is contained in an element of $\mathcal{W}$.*

Proof. By construction, P is a compact subset of a the metric space $\mathbb{R}^m$. Therefore the Lebesgue Covering Lemma implies the existence of a real number $\eta > 0$ such that every subset of diameter $< \eta$ is contained in an element of $\mathcal{W}$. If we choose $r_0 > 0$ such that $r \geq r_0$ implies $\mu(\,\mathbf{B}^r(\mathbf{K})\,) \ < \ \eta$, then $\mathbf{B}^r(\mathbf{K})$ will have the required properties.$\blacksquare$

Homology and barycentric subdivisions

We shall now use the preceding results to show that the homology groups of a barycentric subdivision $B(\mathbf{K})$ are isomorphic to the homology groups of the original complex $\mathbf{K}$. In this case the homology theories will be $H_*(\mathbf{K}^\omega, \mathbf{L}^\omega)$ and $H_*\big(\,B(\mathbf{K})^\tau, B(\mathbf{L})^\tau\,\big)$, and the natural transformation will be associated to maps defined on the chain level. It will suffice to define these chain maps for a simplex and to extend to arbitrary complexes and pairs by putting things together in an obvious manner.

PROPOSITION 11. *Given a nonnegative integer n, let $\partial_j : \Delta_{n-1} \to \Delta_n$ be the order preserving affine map sending Δ_{n-1} to the face of Δ_n opposite the j^{th} vertex, and let $(\delta_j)_\#$ generically denote an associated chain map. Then there are classes $\beta_n \in C_n(\Delta_n^\omega)$ such that β_0 is just the standard generator and if $n > 0$ then*

$$d_n(\beta_n) \quad = \quad \sum_{j=0}^{n} (-1)^j (\partial_j)_\#(\beta_{n-1}) \ .$$

Proof. Since Δ_n is acyclic, it suffices to show that the right hand side lies in the kernel of d_{n-1} if $n > 1$ and in the kernel of ε if $n = 1$. Both of these are routine (but tedious) calculations.■

Using the chains β_n one can piece together chain maps

$$C_*(\mathbf{K}^\omega, \mathbf{L}^\omega) \ \longrightarrow \ C_*\big(\, B(\mathbf{K})^\tau, B(\mathbf{L})^\tau \,\big) \ .$$

We claim these define a natural transformation of homology theories, but in order to do this we must first show that $H_*\big(\, B(\mathbf{K})^\tau, B(\mathbf{L})^\tau \,\big)$ actually defines a homology theory. Properties (a), (c) and (e) follow directly from the construction. Property (b) follows because $B(\Delta_n)$ is star shaped with respect to the vertex $\mathbf{b}$ given by the barycenter of Δ_n. Thus it only remains to verify property (d); in fact, direct inspection similar to an argument in the proof of Theorem 1.6 shows that the map on the chain level is an isomorphism.

By Theorem 1.7, it suffices to check that the natural transformation of homology theories is an isomorphism for a simplicial complex consisting of a single vertex; in fact, for such complexes the map is already an isomorphism on the chain level. Therefore the barycentric subdivision chain maps determine isomorphism of homology groups as asserted in the proposition.■

I.3 : Abstract cell complexes

(Hatcher, Ch. 0)

One possible way to view a polyhedron is to think of it as an object that is constructible in a finite number of steps as follows:

(0) Start with the finite set P_0 of vertices,

(n) If P_{n-1} is the partial polyhedron constructed at Step $(n-1)$, at Step (n) one adds finitely many simplices S_j, identifying each face of each simplex S_j with a simplex in P_{n-1}.

In fact, one can do this in order of increasing dimension, attaching all 1-simplices to the vertices at Step 1, then attaching 2-simplices along the boundary faces at Step 2, and so on. It is often useful in topology to consider objects that are generalizations of this procedure that are more flexible in certain key respects. The objects used these days in algebraic topology are known as **cell complexes**.

One immediate difference between cell complexes and simplicial complexes is that the former use the closed unit disk $D^n \subset \mathbb{R}^n$ and its boundary S^{n-1} in place of an n-simplex Δ and its boundary $\partial\Delta_n$. Since the results of pages 84–85 in `algtop-notes.pdf` (in particular, Theorem VII.1.1) imply that D^n is homeomorphic to Δ_n such that S^{n-1} corresponds to $\partial\Delta_n$, it follows that one can view simplicial complexes as special cases of cell complexes.

Adjoining cells to a space

We shall now give the basic step in the construction of cell complexes. The discussion below relies heavily on the material in Unit V of the online Mathematics 205A notes that were previously cited.

Definition. Let X be a compact Hausdorff space and let A be a closed subset of X. If k is a nonnegative integer, we shall say that *the space X is obtained from A by adjoining finitely many k-cells* if there are continuous mappings $f_i : S^{k-1} \to A$ for $i = 1, \cdots, n$ such that X is homeomorphic to the quotient space of the topological disjoint union

$$A \coprod \left(\{1, \cdots, N\} \times D^k\right)$$

modulo the equivalence relation generated by identifying $(j, \mathbf{x}) \in \{j\} \times S^{k-1}$ with $f_j(\mathbf{x}) \in A$, where the homeomorphism maps $A \subset X$ to the image of A in the quotient by the canonical mapping.

By construction, there is a 1–1 correspondence of sets between X and

$$A \coprod \left(\{1, \cdots, N\} \times \mathbf{open}(D^k)\right)$$

where $\mathbf{open}(D^k) \subset D^k$ is the complement of the boundary sphere. The set $E_j \subset X$ corresponding to the image of $\{j\} \times D^k$ in the quotient is called a *(closed) k-cell*, and the subset $E_j^{\mathbf{O}}$ corresponding to the image of $\{j\} \times \mathbf{open}(D^k)$ in the quotient is called an *open k-cell*. One can then restate the observation in the first sentence of the paragraph to say that X is a union of A and the open k-cells, and these subsets are pairwise disjoint.

Before discussing some topological properties of a space obtained by adjoining k-cells, we shall consider some special cases.

Example 1. Let $(P, \mathbf{K})$ be a simplicial complex,let P_k be the union of all k-simplices in $\mathbf{K}$, and let P_{k-1} be defined similarly. Then the whole point of stating and proving Theorem 1 was to justify an assertion that P_k is obtained from P_{k-1} by attaching k-cells, one for each k-simplex in $\mathbf{K}$. Specifically, for each k-simplex A the map f_A is given by the composite of the homeomorphism $S^{k-1} \to \partial A$ with the inclusion $\partial A \subset P_{k-1}$. The homeomorphism from the quotient of the disjoint union to P_k is given by starting with the composite

$$P_{k-1} \coprod \left(\{1, \cdots, N\} \times D^k\right) \longrightarrow P_{k-1} \amalg_{\partial A} A \longrightarrow P_k$$

where $\amalg_A$ runs over all the k-simplices of $\mathbf{K}$, the first map is a disjoint union of homeomorphisms on the pieces where the maps of Theorem 1 are used to define the homeomorphisms $\{j\} \times D^k \cong A$, and the second map is inclusion on each disjoint summand. This composite passes to a map of the quotient of the space on the left modulo the equivalence relation described above, and it is straightforward to show this map is 1–1 onto and hence a homeomorphism (all relevant spaces are compact Hausdorff).

Example 2. (GRAPHS) As in Section 64 of Munkres, one may define a finite (vertex-edge) graph to be a space obtained from a finite discrete space by adjoining 1-cells. Frequently there is an added condition that the attaching maps for the boundaries should be 1–1 (so that each 1-cell has two endpoints), and the weaker notion introduced in `algtop-notes.pdf` (and Hatcher) is then called a *pseudograph*. The graph corresponds to a simplicial decomposition of a simplicial complex if and only if different 1-cells have different endpoints, and the simplest example of a graph structure that does not come from a simplicial complex is given by taking $X = S^1$ and $A = S^0$ with two 1-cells corresponding to the upper and lower semicircles $E^1_\pm$ in the complex plane. The attaching maps are defined to map the endpoints of $D^1 = [-1, 1]$ bijectively to $-1, 1$. — Another example that is historically noteworthy is the Königsberg Bridge Graph, in which the vertices correspond to four land masses in the city of Königsberg (now Kaliningrad, Russia) and the 1-cells (or *edges*) correspond to the bridges which joined pairs of land masses in the 18^{th} century (see Figure ??? in `advnotesfigures.pdf` for a drawing). This is another example of a graph that does not come from a simplicial complex but is not a pseudograph; if there are two bridges joining the same pairs of land masses, then the graph has two edges with the same boundary points.

Example 3. Yet another example is given by S^n, which is homeomorphic to the quotient D^n/S^{n-1} obtained by identifying all points in the boundary to a single point. An explicit attachment map is given by the continuous onto mapping sending $x \in D^n$ to

$$\left(\frac{x}{2\sqrt{|x| - |x|^2]}}, \; 2|x| - 1 \right) \; ;$$

checking that the first coordinate function is continuous at $x = \mathbf{0}$ and $|x| = 1$ with limits equal to $\mathbf{0}$ is a straightforward exercise (look at the limits as $t \to 0$ and $t \to 1^-$, where t replaces $|x|$ and $\pm\, t$ replaces x). In these examples the attaching maps are constant, which is the complete opposite of being 1–1 for spaces containing more than a single point.

We shall encounter further examples of adjoining cells after we define the main concept of this section. For the time being, we mention a few simple properties of spaces obtained by attaching k-cells for some k

PROPOSITION 2. *If X is obtained from A by attaching 0-cells, then X is homeomorphic to the disjoint union of A with a finite discrete space.*

This is true because the 0-disk D^0 has an empty unit sphere, so there are no attaching maps and the equivalence relation on the space $A \amalg \{1, \cdot, N\}$ is the equality relation.■

PROPOSITION 3. *If X is obtained from A by attaching k-cells, then each open cell $E^{\mathbf{O}}_j$ is an open subset of X, and each such open cell is homeomorphic to* $\mathbf{open}(D^k)$.

Proof. Each closed cell is compact because it is a continuous image of D^k, and hence each such subset is closed in X. By the set-theoretic description given above, the open cell $E^{\mathbf{O}}_j$ is just the complement of the closed set

$$A \; \cup \; \bigcup_{i \neq j} \; E_i$$

and hence it is open in X. Since the quotient space map from the disjoint union to X defines a 1–1 onto continuous mapping from $\mathbf{open}(D^k)$ to $E^{\mathbf{O}}_j$, it suffices to show that an open subset of $\mathbf{open}(D^k)$ is sent to an open subset of $E^{\mathbf{O}}_j$. Let

$$\varphi : A \; \coprod \; \left(\{1, \; \cdots \; , N\} \times D^k \right) \; \longrightarrow \; X$$

be the continuous onto quotient map corresponding to the cell attachments, and suppose that U is open in $\{j\} \times \mathbf{open}(D^k)$. By construction we then have

$$U \quad = \quad \varphi^{-1}\left[\,\varphi[U]\,\right]$$

and thus $\varphi[U]$ is open in X by the definition of the quotient topology.■

The last result in this subsection implies that the inclusion of A in X is homotopically well-behaved if X is obtained from A by adjoining k-cells.

PROPOSITION 4. *If X is obtained from A by attaching k-cells and U is an open subset of X containing A, then there is an open subset V such that*

$$A \ \subset \ V \subset \ \overline{V} \ \subset U$$

and A is a strong deformation retract of both V and $\overline{V}$.

The file `advnotesfigures.pdf` contains an drawing (Figure ???) for the case $N = 1$.

Proof. As in the preceding argument, take

$$\varphi : A \coprod \left(\ \{1, \ \cdots \ , N\} \times D^k \right) \ \longrightarrow \ X$$

to be the continuous onto map corresponding to the k-cell attachments.

Let $F = X - U$, and let $F_0 = \varphi^{-1}[F]$, so that F_0 corresponds to a disjoint union $\amalg_j \ F_j$, where each F_j is a compact subset of $\mathbf{open}(D^k)$; compactness follows because the image of each F_j in X is a closed subset of the compact k-cell E_j. Therefore we can find constants c_j such that $0 < c_j < 1$ and F_j is contained in the open disk of radius c_j about the origin in $\{j\} \times D^k$; let c be the maximum of the numbers c_j, and let $V \subset X$ be the image under φ of the set

$$W \quad = \quad A \coprod \ \left(\bigcup_j \{j\} \times \{ \ \mathbf{x} \in D^k \ | \ c < |\mathbf{x}| \leq 1 \ \} \right) \ .$$

Then V is open because it is the complement of a compact set, and it follows that $\overline{V}$ is the image of

$$Y \quad = \quad A \coprod \ \left(\bigcup_j \{j\} \times \{ \ \mathbf{x} \in D^k \ | \ c \leq |\mathbf{x}| \leq 1 \ \} \right) \ .$$

Each of the sets W and Y is a strong deformation retract of

$$B \quad = \quad A \coprod \ \left(\bigcup_j \{j\} \times S^{k-1} \right) \ .$$

Specifically, the homotopies deforming W and Y into B are the identity on A and map each of the sets $\{ \ c \ < \ |\mathbf{x}| \ \leq \ 1 \ \}$, $\{ \ c \ \leq \ |\mathbf{x}| \ \leq \ 1 \ \}$ to S^{k-1} by sending a (necessarily nonzero) vector $\mathbf{y}$ to $|\mathbf{y}|^{-1}\mathbf{y}$ and taking a staight line homotopy to join these two points. A direct check of the equivalence relation defining φ shows that the associated maps and homotopies $W \to B \to W$ and $Y \to B \to Y$ pass to the quotients $V \to A \to V$ and $\overline{V} \to A \to \overline{V}$, and these quotient maps display A as a strong deformation retract of both V and $\overline{V}$.■

By the preceding discussion, a simplicial complex $(P, \mathbf{K})$ has a finite, linearly ordered chain of closed subspaces

$$\emptyset \;=\; P_{-1} \;\subset\; P_0 \;\subset\; \cdots \;\subset\; P_m \;=\; P$$

such that for each k satisfying $0 \leq k \leq m$, the subspace P_k is obtained from P_{k-1} by attaching finitely many k-cells. We shall generalize this property into a definition for arbitrary cell complex structures.

Definition. Let X be a topological space. A *finite cell complex structure* (or finite CW structure) on X is a chain $\mathcal{E}$ of closed subspaces

$$\emptyset \;=\; X_{-1} \;\subset\; X_0 \;\subset\; \cdots \;\subset\; X_m \;=\; X$$

such that for each k satisfying $0 \leq k \leq m$, the subspace X_k is obtained from X_{k-1} by attaching finitely many k-cells. The subspace X_k is called the *k-skeleton* of X, or more correctly the k-skeleton of $(X, \mathcal{E})$

At this level of abstraction, the notion of cell complex structure is due to J. H. C. Whitehead (1904–1960); his definition extended to infinite cell complex structures and the letters CW were described as abbreviations for two properties of the infinite complexes that are explained in the Appendix of Hatcher's book, but one should also note that the letters also represent Whitehead's last two initials.

It follows immediately that simplicial complexes are examples of cell complexes. Numerous further examples appear on pages 5–8 of Hatcher. Furthermore, the Δ-complexes discussed on pages 102–104 are also examples of cell complexes. In analogy with (edge-vertex) graphs, the main difference between Δ-complexes and simplicial complexes is that two k-simplices in a Δ-complex may have the same faces, but two k-simplices in a simplicial complex have at most a single $(k-1)$-face in common.

Because of the following result, one often describes a cell complex structure as a *cellular decomposition* of X.

PROPOSITION 5. *If X is a space and $\mathcal{E}$ is a cell decomposition of X, then every point of X lies on exactly one open cell of X.*

Proof. Since $X = \cup_k \ (X_k - X_{k-1})$, it follows that every point $y \in X$ lies in a exactly subset of the form $X_k - X_{k-1}$. Therefore there is at most one value of k such that x can lie on an open k-cell. Furthermore, since $X_k - X_{k-1}$ is a union of the open k-cells and the latter are pairwise disjoint, it follows that x lies on exactly one of these open k-cells.■

NOTE. If a cell complex has an n-cell for some $n > 0$ and $0 < m < n$, the cell complex might not have any m-cells (in contrast to the situation for, say, simplicial complexes); see Example 0.3 on page 6 of Hatcher.

Finally, we shall give a slightly different definition of subcomplex than the one in Hatcher.

Definition. If $(X, \mathcal{E})$ is a cell complex, we say that a closed subspace $A \subset X$ determines a *cell subcomplex* if for each $k \geq 0$ the set $A_k = X_k \cap A$ is obtained from A_{k-1} by attaching k-cells such that the every k-cell for A is also a k-cell for X.

There is an simple relationship between this notion of cell subcomplex and the previous definition of subcomplex for a simplicial complex; the proof is straightforward.

PROPOSITION 6. *If $(P, \mathbf{K})$ is a simplicial complex and $(P_1, \mathbf{K}_1)$ is a simplicial subcomplex, then P_1 also determines a cell subcomplex.*■

Finally, here are two further observations regarding subcomplexes. Again, the proofs are straightforward.

PROPOSITION 7. *If X is a cell complex such that $A \subset X$ determines a subcomplex of X and $B \subset A$ determines a subcomplex of A, then B also determines a subcomplex of X. Likewise, if B determines a subcomplex of X then B determines a subcomplex of A.*■

PROPOSITION 8. *If X is a cell complex such that $A \subset X$ determines a subcomplex of X, then for each $k \geq 0$ the set $X_k \cup A$ determines a subcomplex of X.*■

Cellular homology

If P is a polyhedron of positive dimension, the preceding discussion implies that the singular homology groups of P are finitely generated abelian groups. In fact, the conclusion holds more generally if X has the structure of a finite cell complex by the following result:

THEOREM 9. *Let $(X, \mathcal{E})$ be a finite cell complex of dimension n. Then there is a chain complex $\big(C_*(X, \mathcal{E}), d\big)$ such that the chain groups are finitely generated free abelian in every dimension with $C_q(X, \mathcal{E}) = 0$ if $q < 0$ or $q > n$, and the q-dimensional homology of this chain complex is isomorphic to the singular homology group $H_q(X)$.*

The chain complex will be defined explicitly in terms of singular homology and the cell structure for $(X, \mathcal{E})$, and it will be called the *cellular chain complex*. For each k such that $-1 \leq k \leq n$, let X_k denote the k-skeleton of X, where $X_{-1} = \emptyset$. Specifically, we set $C_q(X, \mathcal{E}) = H_q(X_q, X_{q-1})$ and define the differential d_q to be the following composite:

$$H_q(X_q, X_{q-1}) \xrightarrow{\partial[q]} H_{q-1}(X_{q-1}) \xrightarrow{j[q-1]_*} H_{q-1}(X_{q-1}, X_{q-2})$$

These maps define a chain complex since

$$d_{q-1} \circ d_q \quad = \quad j[q-2]_* \circ \partial[q-1] \circ j[q-1]_* \circ \partial[q]$$

and $\partial[q-1] \circ j[q-1]_* = 0$ because the factors are consecutive morphisms in the long exact homology sequence for (X_{q-1}, X_{q-2}). By the results of the preceding section, *the q-dimensional cellular chain group is isomorphic to a free abelian group on the set of q-cells in $\mathcal{E}$.*

Proof of Theorem 9. The result is immediate if $\dim X = 0$ or -1, in which cases X is a nonempty finite set or the empty set. In this case the cellular chain groups are either concentrated in degree zero (the 0-dimensional case) or are all equal to zero (the (-1)-dimensional case).

We shall prove the result for the explicit cellular chain complex described above by induction on $\dim X$, and for this purpose we assume that the result is true when $\dim X \leq n - 1$. The inductive hypothesis then implies that the theorem is true for the $(n-1)$-skeleton X_{n-1}. Now the only difference between the cellular chain complex for X and the corresponding complex for X_{n-1} is that the n-dimensional chain group for the latter is zero while the n-dimensional chain group for the latter is nonzero, and likewise the differentials in both complexes are equal except for the ones going from n-chains to $(n-1)$-chains (in the second case the differential must be zero). It follows that the homology groups of these cell complexes are isomorphic except perhaps in dimensions n and $n - 1$.

Similarly, since $H_q(X_n, X_{n-1}) = 0$ if $q \neq n$ or $n-1$, it follows that $H_q(X) \cong H_q(X_{n-1})$ except perhaps in these dimensions. Therefore, we have shown the inductive step except when $q = n$ or $n-1$. It will be necessary to examine these cases more closely.

We shall describe the n-dimensional homology of $C_*(X, \mathcal{E})$ first. By definition the map d_n is a composite $j[q-1]_* \circ \partial[q]_*$, and the factors fit into the following long exact sequences:

$$0 \;=\; H_n(X_{n-1}) \;\longrightarrow\; H_n(X) \;\longrightarrow\; H_n(X, X_{n-1}) \;\longrightarrow\; H_{n-1}(X_{n-1}) \;\cdots$$

$$0 \;=\; H_{n-1}(X_{n-2}) \;\longrightarrow\; H_{n-1}(X_{n-1}) \;\longrightarrow\; H_{n-1}(X_{n-1} X_{n-2})$$

It follows that $H_n(X)$ is isomorphic to the kernel of $\partial[q]_*$ and the map $j[q-1]_*$ is injective. Similarly, it also follows that $H_{n-1}(X)$ is isomorphic to the kernel of $\partial[q-1]_*$ and the map $j[q-2]_*$ is injective. Since $d_q = j[q-1]_* \circ \partial[q]$, it follows that $H_n(X)$ is also isomorphic to the kernel of d_n, and since $C_{n+1}(X, \mathcal{E}) = 0$ it follows that the kernel of d_n is also isomorphic to the n-dimensional homology of $C_*(X, \mathcal{E})$. Thus we now know the theorem is true for all dimensions except possibly $(n-1)$.

In order to describe the $(n-1)$-dimensional homology of $C_*(X, \mathcal{E})$ we shall consider the following diagram, in which both the row and the column are exact:

$$\begin{array}{ccccccc}
 & & H_{n-1}(X_{n-2}) = 0 & & & & \\
 & & \big\downarrow & & & & \\
\cdots\; H_n(X, X_{n-1}) & \xrightarrow{\partial[n]} & H_{n-1}(X_{n-1}) & \longrightarrow & H_{n-1}(X) & \longrightarrow & H_{n-1}(X, X_{n-1}) = 0 \\
 & & \big\downarrow j[n-1]_* & & & & \\
 & & H_{n-1}(X_{n-1}, X_{n-2}) & & & &
\end{array}$$

By the exactness of the row we know that $H_{n-1}(X)$ is isomorphic to the quotient group

$$H_{n-1}(X_{n-1}) \,/\, \text{Image } \partial[n]$$

and since $j[n-1]_*$ is injective we know from the previous discussion that $j[n-1]_*$ sends $H_{n-1}(X_{n-1})$ onto the kernel of d_{n-1} (note this map is the same for both X and X_{n-1}). Furthermore, by construction we also know that $j[n-1]_*$ maps the image of $\partial[n]$ onto the image of d_n. If we make these substitutions into the displayed expression above, we see that $H_{n-1}(X)$ is isomorphic to the kernel of d_{n-1} modulo the image of d_n, which proves that the conclusion of the theorem also holds in dimension $n-1$.■

If we let $\mathcal{C}(q) = \{ E^q_\alpha \}$ denote the (finite) set of q-cells for $\mathcal{E}$ and view the cellular chain groups $C_q(X, \mathcal{E})$ as free abelian groups on the sets $\mathcal{C}(q)$ by the preceding construction and result, it follows that for each E^q_α we have

$$d_q\,(\,E^q_\alpha\,) \;=\; \sum_{\mathcal{C}(q-1)} [\alpha : \beta]\, E^{q-1}_\beta$$

for suitable integers $[\alpha : \beta]$; classically, these coefficients were called *incidence numbers*. Unlike the situation for simplicial chain complexes, there are no general formulas for finding these numbers. If we already know the homology of X from some other result, then it is often possible to recover them by working backwards (*i.e.*, if we know the homology then often there are not many possibilities for the incidence numbers which will yield the correct homology groups).

One condition under which the incidence numbers are recursively computable is if the cell complex is a **regular cell complex**; in other words, each closed n-cell is in fact homeomorphic to to D^n via the attaching map and is a subcomplex in the evident sense of the word (the boundary is a union of cells in the big complex). These will be true for the cell complexes considered in the next subheading.

Here is a very brief summary of the recursive process: Suppose we have worked out the differentials for the chain complex through dimension $n-1$, and we want to find the differentials in dimension n. Let E be an n-cell; by definition, E determines a cell complex which has the homology of a disk. Let ∂E be the subcomplex given by the boundary, so that we have the incidence numbers on ∂E already. It is only necessary to figure out the map from $\mathbb{Z} = C_n(E)$ to $C_{n-1}(E)$. Now the homology of ∂E is just the homology of S^{n-1}, and since $C_n(\partial E) = 0$ it follows that there are no nontrivial boundaries in $C_{n-1}(\partial E)$, so that $H_{n-1}(\partial E) \cong Z$ may be viewed as a subgroup A of $C_{n-1}(\partial E) = C_{n-1}(E)$. Now the image of this copy of $\mathbb{Z}$ in $C_{n-1}(E)$ represents zero in homology since $H_{n-1}(E) = 0$, and therefore there must be some element in $C_n(E)$ which maps to a generator of A. Since $C_n(E)$ is infinite cyclic, it follows that some multiple of the generator $[E]$ for $C_n(E)$ must map to the generator of A. Let $a \in A$ be the generator such that $d(k[E]) = a$; then it follows that $a = k\,d([E])$. But since $d([E])$ is also a cycle, it follows that $d([E]) = m\,a$ for some integer m. Combining these, we see that $a = km\,a$, and since A is torsion free this implies that $km = 1$, so that $k = m = \pm 1$. Thus we must have $d([E]) = \pm\, a$. the generator of $C_n(E)$. In fact, the exact choice for the sign is unimportant because one obtains the same homology in all cases; we can always choose the generator for $C_n(E)$ so that the incidence number is $+1$. More detailed information is given in the following reference:

> **G. E. Cooke and R. L. Finney.** *Homology of cell complexes (Based on lectures by N. E. Steenrod), Princeton Mathematical Notes No.* 4. *Princeton University Press, Princeton,* 1967.

Convex linear cells

In elementary geometry, the terms *polygon* and *polyhedron* are often used to denote frontiers of bounded open sets in $\mathbb{R}^2$ and $\mathbb{R}^3$ that are defined by finitely many linear equations and inequalities. For example, one has the standard isosceles right triangle in the plane which bounds the compact convex set defined by the inequalities

$$x \geq 0\,, \qquad y \geq 0\,, \qquad x + y \leq 1$$

while standard squares and cubes in the plane and 3-space are defined by

$$0 \leq x,\, y \leq 1\,, \qquad 0 \leq x,\, y,\, z \leq 1$$

and the octagon in the plane with vertices

$$(2, \pm 1),\ (-2, \pm 1),\ (1, \pm 2),\ (-1, \pm 2)$$

is defined by the eight inequalities

$$-2 \leq x,\, y \leq 2\,, \qquad -3 \leq x + y \leq 3\,, \qquad -3 \leq x - y \leq 3\,.$$

Convex sets in $\mathbb{R}^n$ defined by finitely many linear equations and inequalities are basic objects of study in the usual theory of linear programming. In particular, it turns out that the sorts of sets we consider are given by all convex combinations of a finite subset of *extreme points* which correspond to the usual geometric notion of vertices. The reference below is the text for Mathematics 120, which covers linear programming and provides some background on the sets considered here, (particularly in Sections 15.4 – 15.8 on pages 264 – 285).

> **E. K. P. Chong and S. Zak**. An Introduction to Optimization. Wiley, New York, 2001. ISBN: 0-471-39126-3.

We defined convex linear cells in Section I.2; recall that a bounded subset $E \subset \mathbb{R}^n$ is a *convex linear cell* (or also as a *rectilinear cell*) if it is defined by finitely many linear equations and inequalities. It follows immediately that such a set is compact and convex.

The main properties of such cells that we shall need are formulated and proved in Section 7 of [MunkresEDT]. Here is a summary of what we need: If we define a k-plane in a real vector space V to be a set of the form $\mathbf{x} + W$, where W is a k-dimensional vector subspace of V, then the *dimension* of a convex linear cell E is equal to the least k such that E lies in a k-plane. If V is an n-dimensional vector space, this dimension is a nonnegative integer which is less than or equal to n. Suppose now that E is k-dimensional in this sense and $\mathbf{P} = \mathbf{x} + W$ is a k-plane containing E; it follows fairly directly that $\mathbf{P}$ is the unique such k-plane. Less obvious is the fact that the interior of E with respect to $\mathbf{P}$ is nonempty.

> [For the sake of completeness, here is a sketch of the proof: The cell E must contain a set of $k+1$ points that are affinely independent, for otherwise it would lie in a $(k-1)$-plane. Since a convex linear cell is a closed convex set, it must contain the k-simplex whose vertices are these points, and this set has a nonempty interior in the k-plane $\mathbf{P}$.]

It is convenient to describe a minimal and irredundant set of equations and inequalities which define a convex linear cell E. The unique minimal k-plane containing E can be defined as the set of solutions to a system of $n-k$ independent linear equations, and to describe E it is enough to add a MINIMAL set of inequalities which define E.

Definition. If E is a k-dimensional convex linear cell and we are given an efficient set of defining linear equations and inequalities as in the preceding paragraph, then a $(k-1)$-*dimensional face* of E is obtained by taking the subset for which one of the listed inequalities is replaced by an equation.

For example, in the square the four faces are given by adding one of the four conditions

$$x = 0, \qquad x = 1, \qquad y = 0, \qquad y = 1$$

to the equations and inequalities defining the square, and for the 2-simplex whose vertices are $(0,0)$, $(1,0)$ and $(0,1)$ one has the three faces defined by strengthening one of the defining inequalities to one of the three equations $x = 0$, $y = 0$ or $x + y = 1$.

It follows immediately that each $(k-1)$-face of E is a convex linear cell, and Lemmas 7.3 and 7.5 on pages 72 – 74 of [MunkresEDT] show that each face described in this manner is $(k-1)$-dimensional. — One can iterate the process of taking faces and define q-faces of E where $-1 \leq q \leq k$; more details appear on page 75 of the book by Munkres (by definition, the empty set is a (-1)-face).

The geometric boundary of E, written $\mathbf{Bdy}(E)$, may be described in two equivalent ways: It is the union of all the lower dimensional faces ofr E, and it is also the point set theoretic frontier of E in $\mathbf{P}$. We shall need the following theorem, which is discussed on pages 71 – 74 of the Munkres book:

PROPOSITION 10. *If $E \subset \mathbb{R}^n$ is a convex linear cell, then the pair $\big(E, \mathbf{Bdy}(E)\big)$ is homeomorphic to (D^k, S^{k-1}).*

We have already shown this result when E is a simplex by constructing a *radial projection homeomorphism*, and as noted on page 71 of Munkres' book a similar construction proves the corresponding result for an arbitrary convex linear k-cell.■

If we combine this proposition with the remaining material on convex linear cells, we obtain the following basic consequence.

PROPOSITION 11. *If E is a convex linear k-cell and $\mathbf{Bdy}(E)$ is its boundary, then these spaces have cell decompositions such that (i) the cells of $\mathbf{Bdy}(E)$ are the faces of dimension less than k, (ii) the cells of E are the cells of $\mathbf{Bdy}(E)$ together with E itself.*■

If we combine the preceding result with Theorem 3, we obtain the following conclusion relating the geometry and algebraic topology of E and its boundary.

COROLLARY 12. *If E and $\mathbf{Bdy}(E)$ are as above, then there exist chain complexes A_* and B_* such the groups A_q are free abelian groups on the sets of nonempty faces of dimension less than k, the groups B_q are free abelian groups on the sets of nonempty faces of dimension $\leq k$, and the homology groups of A_* and B_* are isomorphic to $H_*(S^{k-1})$ and $H_*(D^k)$ respectively.*■

One can use the preceding discussion to place the proof of Euler's Formula $E + 2 = V + F$ into a more general setting (see the exercises).

I.4 : The Homotopy Extension Property

(Hatcher, Ch. 0, § 2.1)

In this section we shall bring together several concepts from the preceding sections. The basis is the following central Extension Question stated at the beginning of this unit, and our first result describes a condition under which this question always has an affirmative answer.

PROPSITION 1. *Suppose that X and Y are topological spaces, that $A \subset X$ is a retract, and that $g : A \to Y$ is continuous. Then there is an extension of g to a continuous mapping $f : X \to Y$.*

Proof. Let $r : X \to A$ be a continuous function such that $r|A$ is the identity, and define $f = g \circ r$. Then if $a \in A$ we have $f(a) = g \circ r(a) = g(\,r(a)\,)$, and the latter is equal to $g(a)$ because $r|A$ is the identity.■

The hypothesis of the proposition is fairly rigid, but the result itself is a key step in proving a general result on the Extension Question.

THEOREM 2. (HOMOTOPY EXTENSION PROPERTY) *Let $(X, \mathcal{E})$ be a cell complex, and suppose that A determines a subcomplex. Suppose that Y is a topological space, that $g : A \to Y$ is a continuous map, and $f : X \to Y$ is a continuous map such that $f|A$ is homotopic to g. Then there is a continuous map $G : X \to Y$ such that $G|A = g$.*

COROLLARY 3. *Suppose that X and A are as above and that $g : A \to Y$ is homotopic to a constant map. Then g extends to a continuous function from X to Y.*

COROLLARY 4. *Suppose that X and A are as above and that $g : A \to X$ is homotopic to the inclusion map. Then g extends to a continuous function from X to itself.*

Corollary 3 follows because every constant map from A to Y extends to the analogous constant map from X to Y, and Corollary 4 follows because the inclusion of A in X extends continuously to the identity map from X to itself.■

One important step in the proof of the Homotopy Extension Property relies upon the following result:

PROPOSITION 5. *For all $k > 0$ the set $D^k \times \{0\} \cup S^{k-1} \times [0,1]$ is a strong deformation retract of $D^k \times [0,1]$.*

Proof. This argument is outlined in Proposition 0.16 on page 15 of Hatcher, and there is a drawing to illustrate the proof in the file `advnotesfigures.pdf`.

The retraction $r : D^k \times [0,1] \to D^k \times \{0\} \cup S^{k-1} \times [0,1]$ is defined by a radial projection with center $(\mathbf{0}, 2) \in D^k \times \mathbb{R}$. As indicated by the drawing, the formula for r depends upon whether $2|\mathbf{x}| + t \geq 2$ or $2|\mathbf{x}| + t \leq 2$. Specifically, if $2|\mathbf{x}| + t \geq 2$ then

$$r(\mathbf{x},\, t) \quad = \quad \frac{1}{|\mathbf{x}|}\left(\mathbf{x},\, 2|\mathbf{x}| + t - 2\right)$$

while if $2|\mathbf{x}| + t \leq 2$ then we have

$$r(\mathbf{x},\, t) \quad = \quad \frac{1}{2}\left((2-t)\mathbf{x},\, 0\right)$$

and these are consistent when $2|\mathbf{x}| + t = 2$ then both formulas yield the value $|\mathbf{x}|^{-1}(\mathbf{x},\, 0)$. Elementary but slightly tedious calculation then implies that $r(\mathbf{x},\, t)$ always lies in $D^k \times [0,1]$, and likewise that r is the identity on $D^k \times \{0\} \cup S^{k-1} \times [0,1]$.$^{(\star)}$ The homotopy from inclusion $\circ\, r$ to the identity is then the straight line homotopy

$$H(\mathbf{x},\, t;\, s) \quad = \quad (1-s) \cdot r(\mathbf{x},\, t) \; + \; s \cdot (\mathbf{x},\, t)$$

and this completes the proof of the proposition.■

Proof of Theorem 2. In the course of the proof we shall need the following basic fact: *If A and B are compact Hausdorff spaces and $\varphi : A \to B$ is a quotient map in the sense of Munkres' book, then for each compact Hausdorff space C the product map $\varphi \times 1_C : A \times C \to B \times C$ is also a quotient map.* — This follows because $\varphi \times 1_C$ is closed, continuous and surjective; as noted in Exercise 11 on page 186 of Munkres, the same conclusion also holds with weaker hypotheses on φ and C.

Since the homotopy relation on continuous functions is transitive, a standard recursive argument reduces the proof of the theorem to the special cases of subcomplex inclusions

$$X_{k-1} \,\cup\, A \quad \subset \quad X_k \,\cup\, A\,.$$

In other words, *it will suffice to prove the theorem when X is obtained from A by attaching k-cells.*

We now assume the condition in the preceding sentence. Let $h : A \times [0,1] \to Y$ be the homotopy from f (when $t = 0$) to g (when $t = 1$). If we can show that the inclusion

$$A \times [0,1] \,\cup\, X \times \{0\} \quad \subset \quad X \times [0,1]$$

is a retract, then we can use Proposition 1 to find an extension of the map

$$\theta \;=\; \text{“}h\cup f\text{”}\colon\; A\times[0,1]\cup X\times\{0\} \;\longrightarrow\; Y$$

to $X\times[0,1]$, and the restriction of this extension to $X\times\{1\}$ will be a continuous extension of g. — In fact, we shall show that *the space* $A\times[0,1]\cup X\times\{0\}$ *is a strong deformation retract of* $X\times[0,1]$.

As in earlier discussions let

$$\varphi : A \;\amalg\; \left(\{1,\;\cdots\;,N\}\times D^k\right) \;\longrightarrow\; X$$

be the topological quotient map which exists by the definition of attaching k-cells. By Proposition 5 we know that the space

$$A\times[0,1]\;\amalg\;\left(\{1,\;\cdots\;,N\}\right)\times\left(S^{k-1}\times[0,1]\cup D^k\times\{0\}\right)$$

is a strong deformation retract of

$$\left(A\;\amalg\;\{1,\;\cdots\;,N\}\times D^k\;\right)\times[0,1]$$

because we can the mappings piecewise using the identity on $A\times[0,1]$ and the functions from Proposition 5 on each of the pieces $\{j\}\times D^k\times[0,1]$. Let

$$r' : \left(A\;\amalg\;\left(\{1,\;\cdots\;,N\}\times D^k\right)\right)\times[0,1]\;\longrightarrow$$

$$A\times[0,1]\;\amalg\;\left(\{1,\;\cdots\;,N\}\times\left(S^{k-1}\times[0,1]\cup D^k\times\{0\}\right)\right)$$

be the retraction obtained in this fashion, and let

$$H' : \Big(\left(A\;\amalg\;\{1,\;\cdots\;,N\}\times D^k\right)\times[0,1]\Big)\times[0,1]\;\longrightarrow\;\left(A\;\amalg\;\{1,\;\cdots\;,N\}\times D^k\;\right)\times[0,1]$$

be defined similarly. It will suffice to show that these pass to continuous mappings of quotient spaces; in other words, we want to show there are (continuous) mappings r and H such that the following diagrams are commutative, in which ψ is the mapping whose values are given by φ:

$$\begin{array}{ccc}
(A\amalg\cdots)\times[0,1] & \xrightarrow{\;r'\;} & A\times[0,1]\amalg\left(\{1,...,N\}\times[\cdots]\right) \\
\Big\downarrow \varphi\times 1 & & \Big\downarrow \psi \\
X\times[0,1] & \xrightarrow{\;r\;} & A\times[0,1]\cup X\times\{0\}
\end{array}$$

$$\begin{array}{ccc}
\Big((A\amalg\cdots)\times[0,1]\Big)\times[0,1] & \xrightarrow{\;H'\;} & (A\amalg\cdots)\times[0,1] \\
\Big\downarrow \varphi\times 1\times 1 & & \Big\downarrow \phi\times 1 \\
\left(X\times[0,1]\right)\times[0,1] & \xrightarrow{\;H\;} & X\times[0,1]
\end{array}$$

Standard results on factoring maps through quotient spaces imply that such commutative diagrams exist if and only if (i) if two points map to the same point under $\psi\circ r'$, then they map to the same point under $\varphi\times 1$, (ii) if two points map to the same point under $\phi\times 1\circ H'$, then they map to the same point under $\varphi\times 1\times 1$. It is a routine exercise to check both of these statements are true.■

COROLLARY 6. *Suppose that X and Y are as in the theorem and Y is contractible. Then every continuous mapping $f : X\to Y$ has a continuous extension to X.*

Proof. It will suffice to prove that an arbitrary continuous mapping $f : A \to Y$ is homotopic to a constant. We know that 1_Y is homotopic to a constant map k, and therefore $f = 1_Y \circ f$ is homotopic to the constant map $k \circ f$.■

I.5 : Chain homotopies

(Hatcher, § 2.1)

In this section we shall generalize a key step in the proof of that starshaped complexes have acyclic homology. The main feature of the proof is that it constructs an algebraic analog of the straight line contracting homotopy from the identity to the constant map whose value is $\mathbf{v}$.

Definition. Let (A, d) and (B, e) be chain complexes, and let f and g be chain maps from A to B. A *chain homotopy* from f to g is a sequence of mappings $d_k : A_k \to B_{k+1}$ satisfying the following condition for all integers k:

$$d^B_{k+1} \circ D_k \ + \ D_{k-1} \circ d^A_k \ \ = \ \ g_k \ - \ f_k$$

Two chain mappings f, g from A to B are said to be *chain homotopic* if there is a chain homotopy from the first to the second, and this is often written $f \simeq g$.

The proof of the following result is an elementary exercise:

PROPOSITION 1. *The relation $\simeq$ is an equivalence relation on chain maps from one chain complex (A, d) to another (B, e). Furthermore, if f and g are chain homotopic chain maps from (A, d) to (B, e), and h and k are chain homotopic chain maps from (B, e) to (C, θ), then the composites $h \circ f$ and $k \circ g$ are also chain homotopic. Finally, if f, g, h, k are chain maps from A to B and $r \in R$, then $f \simeq g$ and $h \simeq k$ imply $f + h \simeq g + k$ and $rf \simeq rg$.*

Proof. For the first part of the proof let f, g and h be chain maps from (A, d) to (B, e). The zero homomorphisms define a chain homotopy from f to itself. If D is a chain homotopy from f to g then $-D$ is a chain homotopy from g to f. Finally, if D is a chain homotopy from f to g and E is a chain homotopy from g to h, then $D + E$ is a chain homotopy from f to h.

To prove the assertion in the second sentence, let D be a chain homotopy from f to g and let E be a chain homotopy from g to h. Then one can check directly that

$$h \circ D \ + \ E \circ g$$

defines a chain homotopy from $h \circ f$ to $k \circ g$.$^{(\star)}$ The proof of the final assertion is also elementary and is left to the reader.■

Chain homotopies are useful and important because of the following result:

PROPOSITION 2. *If f and g are chain homotopic chain maps from one chain complex (A, d) to another complex (B, e), then the associated homology mappings f_* and g_* are equal.*

Proof. Suppose that $u \in H_k(A)$ and $x \in A_k$ is a cycle representing u, so that $d_k(a) = 0$. If D is a chain homotopy from f to gh, then by definition we have

$$d^B_{k+1} \circ D_k(x) \ + \ D_{k-1} \circ d^A_k \ (x) \ \ = \ \ g_k(x) \ - \ f_k(x)$$

and since $d_k^A(x) = 0$ it follows that the expression above is a boundary. Therefore $g_*(u) - f_*(u)$ must be the zero element of $H_k(B)$.■

An important example

The following basic construction gives an explicit connection between the topological notion of homotopy and the algebraic notion of chain homotopy. Let $n \geq 0$, and let $\mathbf{P}_{n+1}$ denote the *standard* $(n+1)$*-dimensional prism* $\Delta_n \times [0,1]$ with the simplicial decomposition given in Unit II. As in that unit, label the vertices of this prism decomposition by $\mathbf{x}_j = (\mathbf{e}_j, 0)$ and $\mathbf{y}_j = (\mathbf{e}_j, 1)$.

PROPOSITION 3. *The simplicial chain complexes $C_*(\mathbf{P}^\omega_{n+1})$ and $C_*(\mathbf{P}_{n+1})$ are acyclic.*

Proof. These follow from the isomorphism theorem and the fact that $\mathbf{P}_{n+1}$ is star shaped with respect to $\mathbf{y}_n$.■

For each integer j satisfying $0 \leq j \leq n$, let $\partial_j : \Delta_{n-1} \to \Delta_n$ be the affine map which sends Δ_{n-1} to the face opposite the vertex $\mathbf{e}_j$ and is order preserving on the vertices, and let $\partial_j \times \mathbf{I}$ denote the product of the map ∂j with the identity on $[0,1]$. It then follows immediately that we have associated injections of simplicial chain groups

$$(\partial_j)_\# : C_j(\Delta_{n-1}) \longrightarrow C_j(\Delta_n) \ , \qquad (\partial_j \times \mathbf{I})_\# : C_*(\mathbf{P}_{n-1}) \longrightarrow C_*(\mathbf{P}_n)$$

and these are chain maps. Furthermore, these chain maps send ordered chains to ordered chains.

Similarly, for $t = 0, 1$ we also have injections of simplicial chain groups

$$(i_t)_\# : C_*(\Delta_n) \longrightarrow C_*(\mathbf{P}_n)$$

which send a free generator $\mathbf{v}_0 \ \cdots \ \mathbf{v}_q$ to $i_t(\mathbf{v}_0) \ \cdots \ i_t(\mathbf{v}_q)$, where $i_t(\mathbf{v}) = (\mathbf{v}, t)$.

We then have the following result:

THEOREM 4. *For all $n \geq 0$ there are chains $P_{n+1} \in C_{n+1}(\mathbf{P}^\omega_n)$ such that*

$$d_{n+1}(P_{n+1}) \quad = \quad \mathbf{y}_0 \ \cdots \ \mathbf{y}_n \ - \ \mathbf{x}_0 \ \cdots \ \mathbf{x}_n \ - \ \sum_j \ (-1)^j \, (\partial_j \times \mathbf{I})_\#(P_{n-1}) \ .$$

Sketch of proof. Not surprisingly, the construction is inductive, with $P_0 = 0$. Suppose we have constructed the chains P_j for $j \leq n$. There is a chain P_{n+1} with the required properties if and only if the expression on the right hand side of the equation is a cycle, so we need to show that the right hand side vanishes if we apply d_n. This is a straightforward but messy calculation like several previous ones. Some key details are worked out in the bottom half of page 112 of Hatcher.■

The preceding result implies that the inclusion mappings i_t, which are topologically homotopic, determine algebraic chain maps that are chain homotopic. Specifically, if we are given a free generator $\mathbf{v}_0 \ \cdots \ \mathbf{v}_q$ of $C_q(\Delta_n)$ then we may form a chain

$$D_q(\mathbf{v}_0 \ \cdots \ \mathbf{v}_q) \quad \in \quad C_{q+1}(\Delta_n \times \mathbf{I})$$

by substituting $i_0(\mathbf{v})$ for $\mathbf{x}$ and $i_1(\mathbf{v})$ for $\mathbf{y}$. In fact, one can carry out all of this for an arbitrary simplicial complex $(P, \mathbf{K})$, and one has the following conclusion.

PROPOSITION 5. *In the setting above the maps $(i_0)_\#$ and $(i_1)_\#$ from $C_*(\mathbf{K})$ to $C_*(\mathbf{K}\times\mathbf{I})$ are chain homotopic, and hence the associated homology maps*

$$(i_0)_*,\ (i_1)_*\ :\ H_*(\mathbf{K})\ \longrightarrow\ H_*(\mathbf{K}\times\mathbf{I})$$

are equal.■

I.6 : Cones and suspensions

(Hatcher, Ch. 0)

These two basic constructions are described on pages 8–9 of Hatcher. We shall say a little more about them and apply them to construct a homeomorphism from the standard n-disk and $(n-1)$-sphere to the standard n-simplex and its boundary.

The constructions and their properties

Definition. Let X be a topological space. The *cone on* X, usually written $\mathbf{C}(X)$, is the quotient of $X\times[0,1]$ modulo the equivalence relation whose equivalence classes are all one point subsets of the form $\{\ (x,t)\ \}$, where $t\neq 0$, and the subset $X\times\{0\}$.

The first result explains the motivation for the name.

PROPOSITION 1. *If X is a compact subset of $\mathbb{R}^n$, then $\mathbf{C}(X)$ is homeomorphic to a subset of $\mathbb{R}^{n+1}$ so that the image of $X\times\{1\}$ in $\mathbf{C}(X)$ corresponds to $X\times\{0\}$ and every point of the image is on a closed line setment joining a point of the latter to the last unit vector* $(0,\ \cdots\ ,0,1)$.

Proof. Define a continuous map g from $X\times[0,1]$ to $\mathbb{R}^{n+1}$ sending (x,t) to $(tx,1-t)$. This passes to a continuous 1–1 mapping f from $\mathbf{C}(X)$ to $\mathbb{R}^{n+1}$ whose image is the set described in the statement of the result, and since $\mathbf{C}(X)$ is a (continuous image of a) compact space it follows that f maps the cone homeomorphically onto its image.■

Examples. The cone on S^n is canonically homeomorphic to D^{n+1}; specifically, the map $S^n\times[0,1]\to D^{n+1}$ which sends (x,t) to $(1-t)x$ passes to a map of quotients $\mathbf{C}(S^n)\to D^{n+1}$ which is a homeomorphism. Also, the cone on D^n is canonically homeomorphic to D^{n+1}. Perhaps the quickest way to see this is the following: The preceding argument shows that the cone on the upper hemisphere D^n_+ of S^n (where the last coordinate is nonnegative) is the set of points in D^{n+1} whose last coordinate is nonnegative (its "upper half"), so we have to show that the latter is homeomorphic to D^{n+1}. If we let $|x|_2$ and $|x|_\infty$ denote the appropriate norms on $\mathbb{R}^{n+1}$ (see the 205A notes), then the homeomorphism h of $\mathbb{R}^{n+1}$ to itself defined by

$$h(x)\quad=\quad\frac{|x|_\infty}{|x|_2}\cdot x\qquad\text{if}\quad x\ \neq\ 0$$

and $h(0)=0$ (continuity here must be checked, but this is not difficult) will send the upper half of D^{n+1} to the subspace $[-1,1]^n\times[0,1]\subset\mathbb{R}^{n+1}$. Since this product of closed intervals is homeomorphic to $[-1,1]^n$ and the latter is homeomorphic to D^{n+1} by the inverse of the map h, the assertion about $\mathbf{C}(D^n)$ and D^{n+1} follows.■

The cone construction extends to a covariant functor as follows: If $f : X \to Y$ is continuous, then the map $f \times \mathrm{id}_{[0,1]} : X \times [0,1] \to Y \times [0,1]$ is also continuous, and if $q_W : W \times [0,1] \to \mathbf{C}(W)$ is the quotient projection for $W = X$ or Y, then passage to quotients defines a unique continuous mapping $\mathbf{C}(f) : \mathbf{C}(X) \to \mathbf{C}(Y)$ such that

$$\mathbf{C}(f) \circ q_X \quad = \quad q_Y \circ \left(f \times \mathrm{id}_{[0,1]} \right) \ .$$

It is a routine exercise to verify that this construction satisfies the covariant functor identities $\mathbf{C}(\mathrm{id}_{[0,1]}) \ = \ \mathrm{id}_{\mathbf{C}(X)}$ and $\mathbf{C}(g \circ f) \ = \ \mathbf{C}(g) \circ \mathbf{C}(f)$.

Definition. Let X be a topological space. The *(unreduced) suspension on* X, usually written $\mathbf{S}(X)$ or $\Sigma(X)$, is the quotient of $X \times [-1,1]$ modulo the equivalence relation whose equivalence classes are all one point subsets of the form $\{ \ (x,t) \ \}$, where $|t| < 1$, and the subsets $X \times \{\pm\}$.

The suspension of a circle is illustrated in the `figures` file. The name arises because the original space, viewed as the image of $X \times \{0\}$, is effectively "suspended" between the north and south poles (the classes of $X \times \{\pm 1\}$ in the quotient), being held in place by the "cables" $\{x\} \times [-1,1]$.

We have the following analog of Propositions 1 for cones.

PROPOSITION 2. *If X is a compact subset of $\mathbb{R}^n$, then $\mathbf{S}(X)$ is homeomorphic to a subset of $\mathbb{R}^{n+1}$ so that the images of $X \times \{\pm 1\}$ in $\mathbf{S}(X)$ correspond to the point $(0, \ \cdots \ 0, \pm 1)$ and the homeomorphism is the inclusion on $X \times \{0\}$.*

Proof. This is very similar to the proof for cones. Define a continuous map g from $X \times [-1,1]$ to $\mathbb{R}^{n+1}$ sending (x,t) to $\big((1-|t|)x, t \big)$. This passes to a continuous 1–1 mapping f from $\mathbf{S}(X)$ to $\mathbb{R}^{n+1}$ whose image is the set described in the statement of the result, and since $\mathbf{C}(X)$ is a (continuous image of a) compact space it follows that f maps the suspension homeomorphically onto its image.■

Examples. The suspension on S^n is canonically homeomorphic to S^{n+1} by the map sending the class of $(x,t) \in S^n \times [0,1]$ to $\big(\sqrt{1-t^2} \cdot x, t\big) \in \mathbb{R}^{n+1}$. Similarly, the suspension of D^n is canonically homeomorphic to D^{n+1}, and this can be shown by adapting the previous argument which proved that the cone on D^n is homeomorphic to the upper half of D^{n+1} (the cone is just the upper half of the suspension; use symmetry considerations to define the homeomorphism on the lower halves of everything).■

The suspension construction extends to a covariant functor as follows: If $f : X \to Y$ is continuous, then the map $f \times \mathrm{id}_{[-1,1]} : X \times [0,1] \to Y \times [-1,1]$ is also continuous, and if $q_W : W \times [-1,1] \to \mathbf{S}(W)$ is the quotient projection for $W = X$ or Y, then passage to quotients defines a unique continuous mapping $\mathbf{S}(f) : \mathbf{S}(X) \to \mathbf{S}(Y)$ such that

$$\mathbf{S}(f) \circ q_X \quad = \quad q_Y \circ \left(f \times \mathrm{id}_{[0,1]} \right) \ .$$

It is a routine exercise to verify that this construction satisfies the covariant functor identities $\mathbf{S}(\mathrm{id}_{[0,1]}) \ = \ \mathrm{id}_{\mathbf{S}(X)}$ and $\mathbf{S}(g \circ f) \ = \ \mathbf{S}(g) \circ \mathbf{S}(f)$.

Observe that projection onto the second coordinate from $X \times [-1,1]$ to $[-1,1]$ passes to a continuous map from $\mathbf{S}(X)$, and we shall say that the value of this map on a point is the latter's *second coordinate* or *latitude* (the second term is suggested by the drawing in the `figures` file).

Definition. If X is a topological space, then the *upper and lower cones* $\mathbf{C}_{\pm}(X)$ are the subspaces of $\mathbf{S}(X)$ consisting of all classes of all poinst whose second coordinates are nonnegative and nonpositive respectively.

By construction, both the upper and lower cones on X are canonically homeomorphic to the cone on X; in fact, these concepts extend to subfunctors $\mathbf{C}_\pm$ of the suspension functor (in other words, the inclusions of the upper and lower cones are natural transformations).

The exercises for 205B include results showing that the cone or suspension of a polyhedron $P \subset \mathbb{R}^n$ is homeomorphic to a subspace of $\mathbb{R}^{n+1}$ with the following properties;

(1) The unit vector $\mathbf{e}_{n+1}$ is a vertex of the cone, and the unit vectors $\pm\,\mathbf{e}_{n+1}$ are vertices of the suspension.

(2) The intersection of the cone and suspension with $\mathbb{R}^n \times \{0\}$ is just P.

(3) The upper and lower cones of the suspension are the points whose last coordinates are nonnegative and nonpositive respectively.

Homological and homotopic properties of cones and suspensions

The following result is a standard sequence of exercises:

THEOREM 3. *Let X be a topological space (for example, assume X is locally compact Hausdorff or metric).*

(i) The cone $\mathbf{C}(X)$ is contractible to its vertex.

(ii) If X is arcwise connected and locally simply connected, then the suspension $\mathbf{S}(X)$ is simply connected (in fact, the conclusion still holds under weaker assumptions).

(iii) If X is arcwise connected and $q > 0$, then there is a natural isomorphism (in X) from $H_{q+1}(\mathbf{S}(X))$ to $H_q(X)$.■

We shall not prove this, but here are a few hints. First, if $p : X \times [0,1] \to \mathbf{C}(X)$ is the quotient projection, then the product of p with the identity on $[0,1]$ is a quotient map by Exercise 11 on page 186 of Munkres (this is needed to prove that the cone is contractible. The same exercise from Munkres also shows that if we delete either pole from the suspension, then the complement is contractible to the other pole; similarly, the upper cone is a deformation retract of the complement of the south pole and the lower cone is a deformation retract of the complement of the north pole. In particular, the last sentence yields a decomposition of the suspension for whose Mayer-Vietoris sequence is easy to analyze (recall that the complement of both poles is just $X \times (-1,1)$). Finally, the simple connectivity statement is a consequence of the Seifert-van Kampen Theorem.

Finally, here are some remarks about cases not covered in the theorem.

COMPLEMENT TO THEOREM 3. *The suspension is* always *arcwise connected, and there is an isomorphism*

$$H_1(\mathbf{S}(X)) \quad \cong \quad \mathrm{Kernel}\, c_* : H_0(X) \quad \to \quad H_0(\{\mathrm{pt.}\})$$

where c is the constant map.

Arcwise connectedness follows because every point can be joined to the "poles" by a continuous curve, and the statement about H_1 follows from the same Mayer-Vietoris sequence which arises in the proof of Theorem 3.■

II. Construction and uniqueness of singular homology

This unit proves the existence of a homology theory which satisfies nearly all the conditions formulated in Unit VI of `algtop-notes.tex`. The following summarizing table provides more precise references:

Axiom Type	Axiom Numbers	Pages
Primitive Data	(T.1)–(T.5)	74–75
Functoriality and naturality	(A.1)–(A.6)	75–77
Exactness	(B.1)–(B.3)	77–78
Homotopy Invariance	(C.1)	79
Compact/Polyhedral Generation	(C.2)–(C.3)	79–80
Normalization	(D.1)–(D.5)	80–81
Excision	(E.1)–(E.2)	82
Mayer-Vietoris Sequences	(E.3)–(E.4)	82–83

The basic idea of the existence proof is very simple: We modify the construction of simplicial chain complexes to obtain a new functor from the category of topological spaces to the category of chain complexes, and we take the homology groups of these chain complexes. By functoriality, such groups will automatically be topologically invariant. Many steps in verifying the axioms will be fairly straightforward, but there are two crucial pieces of input from Unit I of these notes that will be needed:

(1) In Section I.5 we constructed a chain $P_{q+1} \in C_{q+1}(\Delta_q \times [0,1])$ which was an integral linear combination of all the simplices in $\Delta_q \times [0,1]$ with coefficients ± 1. This chain will be used to show that homotopic maps of spaces define chain homotopic maps of chain complexes, which will imply that the homotopic maps induce the same mappings in homology.

(2) Given an open covering $\mathcal{U}$ of a space X, it is sometimes necessary to know that we can somehow replace an algebraic chain for X by another chain whose pieces are so small that each one lies inside a set in the open covering. If we are dealing with simplicial chains over a simplicial complex, this can be done using iterated barycentric subdivisions. Historically speaking, one of the most important steps in the development of singular homology theory was to "leverage" barycentric subdivision into a construction for singular homology.

In the final section of this unit we shall prove uniqueness theorems for constructions satisfying all the axioms for singular homology described in Unit VI of `algtop-notes.tex` except for (D.5), which relates the fundamental group of an arcwise connected space to its 1-dimensional homology; the statement of this axiom assumes the existence of certain natural transformations relating fundamental groups and homology, and the uniqueness results do not require any of this structure. In Unit III we shall construct these natural transformations from the fundamental group functor to the singular homology theory constructed here, and we shall verify the axiom relating the fundamental group to 1-dimensional homology.

It took about a half century for mathematicians to come up with the formulation that is now standard, starting with Poincaré's initial papers on topology (which he called *analysis situs*) at the end of the 19^{th} century and culminating with the definition of *singular homology* by S. Eilenberg and N. Steenrod in the nineteen forties (with many important contributions by others along the way).

Some books start directly with singular homology and do not bother to develop simplicial homology. The reason for considering the latter here is that it is in some sense a "toy model" of singular homology for which many basic ideas appear in a more simplified framework.

II.1 : Basic definitions and properties

(Hatcher, §§ 2.1, 2.3)

As before, let Δ_q be the standard q-simplex in $\mathbb{R}^{q+1}$ whose vertices are the standard unit vectors $\mathbf{e}_0, \cdots, \mathbf{e}_q$. If $(P, \mathbf{K})$ is a simplicial complex, then for each free generator $\mathbf{v}_0 \cdots \mathbf{v}_q$ of $C_q(P, \mathbf{K})$ there is a unique affine (hence continuous) map $T : \Delta_q \to P$ which sends a point $(t_0, \cdots, t_q) \in \Delta_{q+1}$ to $\sum_j t_j \mathbf{v}_j \in P$. One can think of these as linear simplices in P. The idea of singular homology is to consider more general continuous mappings from Δ_q to a space X, viewing them as simplices with possible singularities or *singular simplices* in the space.

Definition. Let X be a topological space. A *singular q-simplex* in X is a continuous mapping $T : \Delta_q \to X$, and the abelian group of *singular q-chains* $S_q(X)$ is defined to be the free abelian group on the set of singular q-simplices.

If we let $\partial_j : \Delta_{q-1} \to \Delta_q$ be the affine map which sends Δ_{q-1} to the face opposite the vertex $\mathbf{e}_j$ and is order preserving on the vertices, then as in the case of simplicial chains we have boundary homomorphisms $d_q : S_q(X) \to S_{q-1}(X)$ given on generators by the standard formula:

$$d_q(T) = \sum_{j=0}^{n} (-1)^i \partial_i(T) = \sum_{j=0}^{n} (-1)^i T \circ \partial_i$$

Likewise, there are augmentation maps $\varepsilon : S_0(X) \to \mathbb{Z}$ which send each free generator $T : \Delta_0 \to X$ to $1 \in \mathbb{Z}$.

We then have the following results:

PROPOSITION 1. *The homomorphisms d_q make $S_*(X)$ into a chain complex, and if $(P, \mathbf{K})$ is a simplicial complex, then the affine map construction makes $C_*(P, \mathbf{K})$ into a chain subcomplex of $S_q(P)$, and the inclusion is augmentation preserving. Furthermore, if A is a subset of X, then $S_*(A)$ is canonically identified with a subcomplex of $S_*(X)$ by the map taking $T : \Delta_q \to X$ into $i \circ T : \Delta_q \to X$, where $i : A \to X$ is the inclusion mapping.*■

PROPOSITION 2. *Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous map. Then there is a chain map $f_\#$ from $S_*(X)$ to $S_*(Y)$ such that for each singular q-simplex T the value $f_\#(T)$ is given by $f \circ T$. This construction transforms the singular chain complex construction into a covariant functor from topological spaces and continuous maps to chain complexes (and chain maps). Furthermore, passage to quotients yields a covariant functor from pairs of topological spaces and continuous maps of pairs to chain complexes and chain maps.*

This is essentially an elementary verification, and probably the most noteworthy part is the need to verify that $f_\#$ is a chain map. Details are left to the reader.$^{(\star)}$■

Predictably, the homology groups we want are the homology groups of the singular chain complexes.

Definition. If X is a topological space, then the *singular homology groups* $H_*(X)$ are the corresponding homology groups of the chain complex defined by $S_*(X)$. More generally, if A is a subset of X, then the *relative chain complex* $S_*(X, A)$ is defined to be $S_*(X)/S_*(A)$, and the *relative singular homology groups* $H_*(X, A)$ are the corresponding homology groups of that quotient complex. Note that if $(\mathbf{K}, \mathbf{L})$ is a pair consisting of a simplicial complex and a subcomplex with underlying space pair (P, Q), then Proposition 1 generalizes to yield a chain map from $\theta_\# : C_*(\mathbf{K}, \mathbf{L})$ to $S_*(P, Q)$. — Note that the relative groups (both singular and simplicial) do not have augmentation homomorphisms if A or $\mathbf{L}$ is nonempty.

It is not difficult to show that the singular homology groups of homeomorphic spaces are isomorphic, and in fact it is an immediate consequence of the following results:

PROPOSITION 3. *The homology groups $H_*(X, A)$ and homomorphisms $f_*; H_*(X, A) \to H_*(Y, B)$ define a covariant functor from the category of pairs of topological spaces to the category of abelian groups and homomorphisms. Furthermore, if $(\mathbf{K}, \mathbf{L})$ is a pair consisting of a simplicial complex and a subcomplex with underlying space pair (P, Q), then the chain map $\theta_\#$ induces a natural transformation of functors $\theta_* : H_*(\mathbf{K}, \mathbf{L}) \to H_*(P, Q)$.*■

This proposition shows that we have data types (T.3) and (T.5) in our axiomatic description of singular homology, and it also verifies axioms (A.1) and (A.2), which involve functoriality and naturality with respect to simplicial homology.

Since functors send isomorphisms in source category to isomorphisms in the target, the topological invariance of singular homology groups is a trivial consequence of Proposition 3.

COROLLARY 4. *If X and Y are topological spaces and $f : X \to Y$ is a homeomorphism, then the associated homomorphism of graded homology groups $f_* : H_*(X) \to H_*(Y)$ is an isomorphism.*■

By Corollary 3, the simplicial homology groups of homeomorphic polyhedra will be isomorphic if we can give an affirmative answer to the following question for all simplicial complexes $(P, \mathbf{K})$:

PROBLEM. If $(P, \mathbf{K})$ is a simplicial complex and $\lambda : C_*(\mathbf{K}) \to S_*(P)$ is the associated chain map, does $\theta_* : H_*(\mathbf{K}) \to H_*(P)$ define an isomorphism of homology groups?

We shall prove this later. For the time being we note that *the map λ is a chain level isomorphism if $\mathbf{K}$ is given by a single vertex* (in this case each of the groups $S_q(X)$ is cyclic, and it is generated by the constant map from Δ_q to X).

The simplest normalization properties of homology groups

It will be convenient to go through the verifications roughly in order of increasing complexity rather than to follow the ordering given in `algtop-notes.pdf`. From this viewpoint, the next axioms to consider are the normalization axioms (D.2)–(D.4); it is mildly ironic that (D.1) will be one of the last axioms to be verified.

The verification of (D.4), which states that negative-dimensional homology groups are zero, is particularly tirival; the simplicial chain groups $S_q(X, A)$ vanish by construction if $q < 0$, and since the homology groups are subquotients of the chain groups they must also vanish.

If X is a topological space and $T : \Delta_q \to X$ is a singular simplex, then the image of T lies entirely in a single path component of X. Therefore the next result, whose conclusion includes the statement of (D.2), follows immediately.

PROPOSITION 5. *If X is a topological space and its path components are the subspaces X_α, then the maps $S_*(X_\alpha)$ to $S_*(X)$ induced by inclusion define an isomorphism of chain complexes $\bigoplus S_*(X_\alpha) \to S_*(X)$ and hence also a homology isomorphism from $\bigoplus H_*(X_\alpha)$ to $H_*(X)$.*■

The preceding results lead directly to a verification of (D.3).

COROLLARY 6. *In the setting above, $H_0(X)$ is isomorphic to the free abelian group on the set of path components of X.*

A proof of this result is given on pages 109 – 110 of Hatcher.■

One immediate consequence of the preceding observations is that the map from $C_*(\mathbf{K})$ to $S_*(P)$ is an isomorphism if $(P, \mathbf{K})$ is 0-dimensional, and similarly for the map from $H_*(\mathbf{K})$ to $H_*(P)$.

Although we are far from ready to verify (D.1) in complete generality, we can do so for the very simplest examples.

PROPOSITION 7. (The Eilenberg-Steenrod Dimension Axiom) *If $X = \{x\}$ consists of a single point, then $H_q(X) = 0$ if $q \neq 0$, and $H_0(X) \cong \mathbb{Z}$ with the isomorphism given by the augmentation map.*

Proof. Suppose first that $x \in \mathbb{R}^n$ for some n, so that $\{x\}$ is naturally a 0-dimensional polyhedron. We have already noted that the simplicial and singular chains on X are isomorphic. Since the conclusion of the proposition holds for (unordered) simplicial chains by the results of the preceding unit, it follows that the same holds for singular chains. To prove the general case, note that if $\{x\}$ is an arbitrary space consisting of a single point and $\mathbf{0} \in \mathbb{R}^n$, then $\{\mathbf{0}\}$ is homeomorphic to $\{x\}$ and in this case the conclusion follows from the special case because homeomorphic spaces have isomorphic homology groups.■

The compact supports property

Our next result verifies (C.2) and is often summarized with the phrase, *singular homology is compactly supported*. This was not one of the original Eilenberg-Steenrod axioms, but its importance for using singular homology was already clear when Eilenberg and Steenrod developed singular homology.

THEOREM 8. *Let X be a topological space, and let $u \in H_q(X)$. Then there is a compact subspace $A \subset X$ such that u lies in the image of the associated map from $H_q(A)$ to $H_q(X)$. Furthermore, if A is a compact subset of X and $u, v \in H_q(A)$ are two classes with the same image in $H_q(X)$, then there is a compact subset B satisfying $A \subset B \subset X$ such that the images of u and v are equal in $H_q(B)$.*

Proof. If c is a singular q-chain and

$$c \;=\; \sum_j n_j\, T_j$$

define the *support* of c, written $\mathsf{Supp}\,(c)$, to be the compact set $\cup_j\ T_j(\Delta_q)$. Note that this subset is compact.

If $u \in H_q(X)$ is represented by the chain z and if $A = \mathsf{Supp}\,(z)$, then since $S_*(A) \to S_*(X)$ is 1–1 it follows that z represents a cycle in A and hence u lies in the image of $H_q(A) \to H_q(X)$.

Suppose now that A is a compact subset of X and $u, v \in H_q(A)$ are two classes with the same image in $H_q(X)$. Let z and w be chains in $S_q(A)$ representing u and v respectively, and let $b \in S_{q+1}(X)$ be such that $d(b) = i_\#(z) - i_\#(w)$. If we set $B = A \cup \mathsf{Supp}\,(b)$, then it follows that the images of $z - w$ bounds in $S_q(B)$, and therefore it follows that u and v have the same image in $H_q(B)$.■

II.2 : Exactness and homotopy invariance

(Hatcher, §§ 2.1, 2.3)

We have seen that long exact sequences and homotopy invariance yield a great deal of information about homology groups. The next step is to verify some of the properties for singular homology and their compatibility with the analogous properties for simplicial homology.

The exact sequence of a pair

In 205B the long exact sequence of a pair in simplicial homology turned out to be a direct consequence of the corresponding long exact homology sequence for a short exact sequence of chain complexes. In view of our definitions, it is not surprising that the same considerations yield long exact sequences of pairs in singular homology.

THEOREM 1. (Long Exact Homology Sequence Theorem — Singular Homology Version). *Let (X, A) be a pair of topological spaces where A is a subspace of X. Then there is a long exact sequence of homology groups as follows:*

$$\cdots \quad H_{k+1}(X,A) \xrightarrow{\partial} H_k(A) \xrightarrow{i_*} H_k(X) \xrightarrow{j_*} H_k(X,A) \xrightarrow{\partial} H_{k-1}(A) \quad \cdots$$

This sequence extends indefinitely to the left and right. Furthermore, if we are given another pair of spaces (Y, B) and a continuous map of pairs $f : (X, A) \to (Y, B)$ such that $f : X \to Y$ is continuous and $f[A] \subset B$, then we have the following commutative diagram in which the two rows are exact:

$$\begin{array}{ccccccccccc} \cdots & H_{k+1}(X,A) & \xrightarrow{\partial} & H_k(A) & \xrightarrow{i_*} & H_k(X) & \xrightarrow{j_*} & H_k(X,A) & \xrightarrow{\partial} & H_{k-1}(A) & \cdots \\ \cdots & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & \\ \cdots & H_{k+1}(Y,B) & \xrightarrow{\partial'} & H_k(B) & \xrightarrow{i'_*} & H_k(Y) & \xrightarrow{j'_*} & H_k(Y,B) & \xrightarrow{\partial'} & H_{k-1}(B) & \cdots \end{array}$$

This follows immediately from the algebraic theorem on long exact homology sequences and the definitions of the various homology groups in terms of a short exact sequence of chain complexes.■

There is also a map of long exact sequences relating simplicial and singular homology for simplicial complexes. This is not one of the Eilenberg-Steenrod properties, but logically it fits naturally into the discussion here.

THEOREM 2. *Let $(X, \mathbf{K})$ be a simplicial complex, and let $(A, \mathbf{L})$ determine a subcomplex. Then there is a commutative ladder as below in which the horizontal lines represent the long exact homology sequences of pairs and the vertical maps are the natural transformations from simplicial to singular homology.*

$$\begin{array}{ccccccccccc} \cdots & H_{k+1}(\mathbf{K},\mathbf{L}) & \xrightarrow{\partial} & H_k(\mathbf{L}) & \xrightarrow{i_*} & H_k(\mathbf{K}) & \xrightarrow{j_*} & H_k(\mathbf{K},\mathbf{L}) & \xrightarrow{\partial} & H_{k-1}(\mathbf{L}) & \cdots \\ \cdots & \downarrow \lambda_* & & \downarrow \lambda_* & & \downarrow \lambda_* & & \downarrow \lambda_* & & \downarrow \lambda_* & \\ \cdots & H_{k+1}(X,A) & \xrightarrow{\partial} & H_k(A) & \xrightarrow{i_*} & H_k(X) & \xrightarrow{j_*} & H_k(X,A) & \xrightarrow{\partial} & H_{k-1}(A) & \cdots \end{array}$$

The results follow directly from the Five Lemma and the fact that the previously defined chain maps λ pass to morphisms of quotient complexes of relative chains from $C_*(\mathbf{K},\mathbf{L})$ to $S_*(X, A)$.■

Theorems 1 and 2 combine to show that our construction has several of the necessary properties for an abstract singular homology theory; namely, it yields data types (T.2) and (T.5) and axioms (A.2)–(A.3), (A.5) and (B-1)–(B.3). The remainder of this section is devoted to verifying axiom (C.1), and thus the results of this section reduce the verification of singular homology axioms to the following:

(1) Construction of data type (T.2).

(2) Verification of axioms (A.4), (D.1) and (E.1)–(E.4).

(3) Construction of data type (T.4), and verification of axioms (A.6), (C.3) and (D.5).

We shall take care of the first two points in Sections II.3 and II.4. This will prove that one has a theory with all the properties needed to derive the applications in Unit VII in `algtop-notes.pdf`. Axiom (C.3) will be needed to prove the uniqueness results for axiomatic singular homology in Section II.5, and a reader who wishes to skip this may do so without loss of continuity. Finally, data type (T.4), and axioms (A.6) and (D.5) are not needed to prove uniqueness, and we are postponing the discussion of these features until the next unit.

Homotopy invariance

By definition, two maps of topological space pairs $f, g : (X, A) \to (Y, B)$ are homotopic *as maps of pairs* if there is a homotopy $H : (X \times [0,1], A \times [0,1]) \to (Y, B)$ such that the restriction of H to $(X \times \{0\}, A \times \{0\})$ and $(X \times \{1\}, A \times \{1\})$ are given by f and g respectively

The discussion of chain homotopies in Section I.5 suggests the following question: *If f and g are homotopic maps from (X, A) to (Y, B), will the associated chain maps from $S_q(X, A)$ to $S_q(Y, B)$ be chain homotopic?*

An affirmative answer to this question implies axiom (C.1), which states that homotopic maps of pairs induce the same homomorphisms in singular homology. The next result confirms that the answer to the preceding question is yes.

THEOREM 3. (Homotopy invariance of singular homology) *Suppose that $f, g : (X, A) \to (Y, B)$ are homotopic as maps of pairs. Then the associated chain maps $f_\#, g_\# : S_*(X, A) \to s_*(Y, B)$ are chain homotopic, and the associated homology homomorphisms $f_*, g_* : H_*(X, A) \to H_*(Y, B)$ are equal.*

Before proving this result, we shall state three important consequences.

COROLLARY 4. *If $f : X \to Y$ is a homotopy equivalence, then the associated homology maps $f_* : H_*(X) \to H_*(Y)$ are isomorphisms.*

Proof. Let $g : Y \to X$ be a homotopy inverse to f. Since $g \circ f$ is homotopic to the identity on X and $g \circ g$ is homotopic to the identity on Y, it follows that the composites of the homology maps $g_* \circ f_*$ and $f_* \circ g_*$ are equal to the identity maps on $H_*(X)$ and $H_*(Y)$ respectively, and therefore f_* and g_* are isomorphisms.■

COROLLARY 5. *If X is a contractible space and there is a contracting homotopy from the identity to the constant map whose value is given by $y \in X$, then the inclusion of $\{y\}$ in X defines an isomorphism of singular homology groups.*

Proof. Let $i : \{y\} \to X$ be the inclusion map, and let $r : X \to \{y\}$ be the constant map, so that $r \circ i$ is the identity. The contracting homotopy is in fact a homotopy from the identity to the reverse composite $i \circ r$, and therefore $\{y\}$ is a deformation retract of X. By the preceding corollary, it follows that i_* defines an isomorphism of singular homology groups.■

COROLLARY 6. *If $f : (X, A) \to (Y, B)$ is a continuous map of pairs such that the associated maps $X \to Y$ and $A \to B$ are homotopy equivalences, then the homology maps f_* from $H_*(X, A)$ to $H_*(Y, B)$ all isomorphisms.*

Proof. In this case we have a commutative ladder as in Theorem 1, in which the horizontal lines represent the exact homology sequences of (X, A) and (Y, B), while the vertical arrows represent the homology maps defined by the mapping f. Since the mappings from X to Y and from A to B are homotopy equivalences, it follows that all the vertical maps except possibly those involving $H_*(X, A) \to H_*(Y, B)$ are isomorphisms; one can now use the Five Lemma to prove that these remaining vertical maps are also isomorphisms.■

The following simple observation will be useful in the proof of Theorem 3:

LEMMA 7. *For each $t \in [0, 1]$ let $i_t : X \to X \times [0, 1]$ denote the slice inclusion $i_t(x) = (x, t)$, Then i_0 and i_1 are homotopic.*

Proof. The identity map on $X \times [0, 1]$ defines a homotopy from i_0 to i_1.■

Proof of Theorem 3. We shall first show that it suffices to prove the theorem for the mappings i_0 and i_1 described in Lemma 7. For suppose we have continuous mappings $f, g : X \to Y$ and a homotopy $H : X \times [0, 1] \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$. Then we also have

$$f_* \;=\; (H \circ i_0)_* \;=\; H_* \circ (i_0)_* \;=\; H_* \circ (i_1)_* \;=\; (H \circ i_1)_* \;=\; g_*$$

showing that f and g define the same maps in homology.

To prove the result for the mappings in Lemma 7 we shall in fact prove that *the chain maps $(i_0)_\#$ and $(i_1)_\#$ from $S_*(X)$ to $S_*(X \times [0, 1])$ are chain homotopic.* — The results of Section I.5 will then imply that the homology maps $(i_0)_*$ and $(i_1)_*$ are equal.

In Section I.5 we noted the existence of simplicial chains $P_{q+1} \in C_{q+1}(\Delta_q \times [0, 1])$ such that $P_0 = 0$, $P_1 = \mathbf{y}_0\mathbf{x}_0$ and more generally

$$dP_{q+1} \;=\; (i_1)_\# \mathbf{1}_q \;-\; (i_0)_\# \mathbf{1}_q \;-\; \sum_j (-1)^j (\partial_j \times 1)_\# P_q$$

where $\mathbf{1}_q = \mathbf{e_0} \cdots \mathbf{e}_q \in C_q(\Delta_q)$, the map $\partial_j = \partial_j^{[q]} : \Delta_{q-1} \to \Delta_q$ is affine linear onto the face opposite $\mathbf{e}_j$, and $(-)_\#$ generically denotes an associated chain map. Recall that the existence of the chains P_{q+1} was proved inductively, the key point being that since $\Delta_q \times \mathbf{I}$ is acyclic, such a chain exists if the boundary of

$$(i_1)_\# \mathbf{1}_q \;-\; (i_0)_\# \mathbf{1}_q \;-\; \sum_j (-1)^j (\partial_j \times 1)_\# P_q$$

is equal to zero.

To construct the chain homotopy $K : S_q(X) \to S_{q+1}(X \times [0, 1]$, let $T : \Delta_q \to X$ be a free generator of $S_q(X)$ and set $K(T) = (T \times \mathrm{id}_{[0,1]})_\# P_{q+1}$. We then have

$$d\,K(T) \;=\; d \circ (T \times \mathrm{id}_{[0,1]})_\# P_{q+1} \;=\; (T \times \mathrm{id}_{[0,1]})_\# \circ d(P_{q+1}) \;=\;$$

$$(T \times 1)_{\#} \circ (i_1)_{\#} \mathbf{1}_q \; - \; (T \times 1)_{\#} \circ (i_0)_{\#} \mathbf{1}_q \; - \; \sum_j (-1)^j d \circ (T \circ \partial_j \times 1)_{\#} P_q \;\; =$$

$$(i_1)_{\#} \circ T_{\#}(\mathbf{1}_q) \; - \; (i_0)_{\#} \circ T_{\#}(\mathbf{1}_q) \; - \; \sum_j (-1)^j (T \circ \partial_j \times 1)_{\#} d(P_q) \;\; =$$

$$(i_1)_{\#}(T) \; - \; (i_0)_{\#}(T) \; - \; K \circ d(T) \; .$$

Therefore K defines a chain homotopy between $(i_1)_{\#}$ and $(i_0)_{\#}$ as required.■

II.3 : Excision and Mayer-Vietoris sequences

(Hatcher, §§ 2.1 – 2.3)

The final Eilenberg-Steenrod axiom, called *excision*, is the most complicated to state and to prove, and the crucial steps in the argument trace back to the proofs of the following two results in simplicial homology theory:

(1) If the polyhedron P is obtained from the polyhedron Q by adjoining a single simplex S (whose boundary must lie in Q), then the inclusion from $(S, \partial S)$ to (P, Q) defines an isomorphism in simplicial homology. More generally, if P_1 and P_2 correspond to subcomplexes of P in some simplicial decomposition and $P = P_1 \cup P_2$, then the inclusion map from $(P_1, P_1 \cap P_2)$ to $(P = P_1 \cup P_2, P_2)$ defines isomorphisms in homology.

(2) For every simplicial complex $(P, \mathbf{K})$, the homology groups of $(P, \mathbf{K})$ and its barycentric subdivision $\big(P, B(\mathbf{K})\big)$ are naturally isomorphic (with respect to subcomplex inclusions).

In particular, the excision axioms are essentially abstract, highly generalized versions of statement (1), both in terms of their formulations and their proofs. Usually the following restatement of (E.2) is taken to be the main version of excision.

THEOREM 1. *Suppose that (X, A) is a topological space and that U is a subset of X such that $U \subset \overline{U} \subset$ Interior(A). Then the inclusion map from $(X - U, A - U)$ to (X, A) determines isomorphisms in homology.*

Here is the analogous restatement of (E.1).

THEOREM 2. *Suppose that the space X can be written as a union of subsets $A \cup B$ such that the interiors of A and B form an open covering of X. Then the inclusion of pairs from $(B, A \cap B)$ to $(X = A \cup B, A)$ induces isomorphisms in homology.*

In particular, *the conclusion of Theorem 2 is valid if both A and B are open subsets of X.*

One can derive Theorem 1 as a consequence of Theorem 2 by taking $B = X - U$ (note that the open set $X - \overline{U}$ is contained in $X - U$).■

There is an obvious formal similarity involving the most general statement in (1), the statement of (E.1) in Theorem 2, and the standard module isomorphism

$$M/M \cap N \;\; \cong \;\; M + N/N \; x \quad \text{(where } \; M \text{ and } N \; \text{ are submodules of some module } \; L)$$

and we shall see that the similarities are more than just a coincidence.

Using the acyclicity of $C_*(\Delta_q)$ we may inductively construct chains $\beta_q \in C_q\big(B(\Delta_q)\big)$ (simplicial chains on the barycentric subdivision) such that $\beta_0 = \mathbf{1}_0$ and

$$d(\beta_q) \;=\; \sum_j (-1)^j\,(\partial_j)_\#\beta_{q-1}$$

for $q \geq 0$. If X is a topological space, then we may define a graded homomorphism $\beta : S_*(X) \to S_*(X)$ such that for each singular simplex $T : \Delta_q \to X$ we have $\beta(T) = T_\#(\beta_q)$.

LEMMA 3. *The graded homomorphism β is a map of chain complexes. Furthermore, if A is a subspace of X then β maps $S_*(A)$ into itself.*

Proof. We have $d\circ\beta(T) = d\circ T_\#(\beta_q) = T_\#\circ d(\beta_q)$, and the last term is equal to

$$T_\#\left(\sum_j (-1)^j\,(\partial_j)_\#\beta_{q-1}\right) \;=\; \sum_j (-1)^j\,(T\circ\partial_j)_\#\beta_{q-1}$$

which in turn is equal to $\beta\big(d(T)\big)$.■

The significance of the barycentric subdivision chain map is that it takes a chain in a given homology class and replaces it by a chain which is in the same homology class but is composed of smaller pieces; in fact, if one applies barycentric subdivision sufficiently many times, it is possible to find a chain representing the same homology class such that its chain are arbitrarily small. Justifications of these assertions will require several steps.

The first objective is to prove that the barycentric subdivision map is chain homotopic to the identity. As in previous constructions, this begins with the description of some universal examples.

PROPOSITION 4. *There are singular chains $L_{q+1} \in S_{q+1}(\Delta_n)$ such that $L_1 = 0$ and $d(L_{q+1}) = \beta_q - \mathbf{1}_q - \sum_j (-1)^j(\partial_j)_\#(L_q)$.*

By convention we take $L_0 = 0$.

Sketch of proof. Once again, the idea is to construct the chains recursively. Since Δ_q is acyclic, we can find a chain with the desired properties provided the difference

$$\beta_q \;-\; \mathbf{1}_q \;-\; \sum_j (-1)^j(\partial_j)_\#(L_q)$$

is a cycle. One can prove this chain lies in the kernel of d_q by using the recursive formulas for $d_q(\beta_q)$, $d_q(\mathbf{1}_q)$, and $d_q(L_q)$.$^{(\star)}$■

PROPOSITION 5. *If X is a topological space and $A \subset X$ is a subspace, then the identity and the barycentric subdivision maps on $S_*(X,A)$ are chain homotopic.*

Proof. It will suffice to construct a chain homotopy on $S_*(X)$ that sends the subcomplex $S_*(A)$ to itself, for one can then obtain the relative statement by passage to quotients.

Define homomorphisms $W : S_q(X) \to S_{q+1}(X)$ on the standard free generators $T : \Delta_q \to X$ by the formula

$$W(T) \;=\; T_\# L_{q+1}\ .$$

By construction, if $T \in S_q(A)$ then $W(T) \in S_{q+1}(A)$. The proof that W is a chain homotopy uses the recursive formula for L_{q+1} and is entirely analogous to the proof that the map K in the proof of Theorem ????? is a chain homotopy.■

Small singular chains

We have noted that barycentric subdivision takes a cycle and replaces it by a homologous cycle composed of smaller pieces and that if one iterates this procedure then one obtains a chain whose pieces are arbitrarily small. Not surprisingly, we need to formulate this more precisely.

Definition. Let X be a topological space, and let $\mathcal{F}$ be a family of subsets whose interiors form an open covering of X. A singular chain $\sum_i \, n_i \, T_i \in S_q(X)$ is said to be *$\mathcal{F}$-small* if for each i the image $T_i(\Delta_q)$ lies in a member of $\mathcal{F}$. Denote the subgroup of $\mathcal{F}$-small singular chains by $S_*^{\mathcal{F}}(X)$. It follows immediately that the latter is a chain subcomplex of $S_*^{\mathcal{F}}(X)$; furthermore, if $A \subset X$ and we define $S_*^{\mathcal{F}}(A)$ to be the intersection of $S_*^{\mathcal{F}}(X)$ and $S_*^{\mathcal{F}}(A)$, then we may define relative $\mathcal{F}$-small chain groups of the form

$$S_*^{\mathcal{F}}(X,A) \quad = \quad S_*^{\mathcal{F}}(X)/S_*^{\mathcal{F}}(A) \ .$$

Note further that the barycentric subdivision maps send $\mathcal{F}$-small chains into $\mathcal{F}$-small chains.

THEOREM 6. *For all (X,A) and $\mathcal{F}$, the inclusion mappings $S_*^{\mathcal{F}}(X,A) \to S_*(X,A)$ define isomorphisms in homology.*

Proof. It is a straightforward algebraic exercise to prove that if L is a chain homotopy from the barycentric subdivision map β to the identity, then for each $r \geq 1$ the map $(1 + \ \cdots \ + \beta^{r-1}) \circ L$ defines a chain homotopy from β^r to the identity.

Let $\mathcal{U}$ be the open covering of X obtained by taking the interiors of the sets in $\mathcal{F}$.

Suppose first that we have $u \in H_*(X,A)$ and u is represented by the cycle $z \in S_*(X,A)$. Write $z = \sum_i \, n_i T_i$ and construct open coverings $\mathcal{W}_i$ of Δ_q by $\mathcal{W}_i = T_i^{-1}(\Delta_q)$. Then by the Lebesgue Covering Lemma there is a positive integer r such that for each i, every simplex in the r^{th} barycentric subdivision of Δ_q lies in a member of $\mathcal{W}_i$. It follows immediately that $\beta^r(z)$ is $\mathcal{F}$-small. Since β^r is a chain map, it follows that $\beta^r(z)$ is also a cycle in both $S_*(X,A)$ and the subcomplex $S_*^{\mathcal{F}}(X,A)$, and since β is chain homotopic to the identity it follows that

$$u \quad = \quad \beta_*(u) \quad = \quad \cdots \quad = \quad (\beta_*)^r(u) \quad = \quad (\beta^r)_*(u)$$

and hence u lies in the image of the homology of the small singular chain group.

To complete the proof we must show that if two cycles in $S_*^{\mathcal{F}}(X,A)$ are homologous in $S_*(X,A)$ then they are also homologous in $S_*^{\mathcal{F}}(X,A)$. Let z_1 and z_2 be the cycles, and let $dw = z_2 - z_1$ in $S_*(X,A)$. As in the preceding paragraph there is some t such that $\beta^t(w) \in S_*^{\mathcal{F}}(X,A)$. Since β^t is a chain map and is chain homotopic to the identity, it follows that we have

$$[z_2] \quad = \quad (\beta^t)_*[z_2] \quad = \quad [\beta^t(z_2)] \quad = \quad [\beta^t(z_1)] \quad = \quad (\beta^t)_*[z_1] \quad = \quad [z_1]$$

in the $\mathcal{F}$-small homology $H_*^{\mathcal{F}}(X,A)$. Therefore we have shown that the map from $H_*^{\mathcal{F}}(X,A)$ to $H_*(X,A)$ is also injective, and hence it must be an isomorphism.■

Application to Excision

We recall the hypotheses of the Excision Property: A pair of topological spaces (X, A) is given, and we have an open subset $U \subset X$ such that $\overline{U} \subset \mathrm{Int}(A)$. Excision then states that the inclusion map of pairs from $(X - U, A - U)$ to (X, A) defines isomorphisms of singular homology groups.

Predictably, we shall use the previous results on small chains. Let $\mathcal{F}$ be the family of subsets given by A and $X - U$. Then by the hypotheses we know that the interiors of the sets in $\mathcal{F}$ form an open covering of X, and by definition the subcomplex $S_*^{\mathcal{F}}(X)$ is equal to $S_*(A) + S_*(X - U)$. Therefore the chain level inclusion map from $S_*(X - U, A - U)$ to $S_*(X, A)$ may be factored as follows:

$$S_*(X-U, A-U) \quad = \quad S_*(X-U)/S_*(A-U) \quad = \quad S_*(X-U)/\left(\, S_*(A) \cap S_*(X-U)\,\right) \quad \longrightarrow$$

$$\left(\, S_*(A) + S_*(X-U)\,\right)/S_*(A) \quad = \quad S_*^{\mathcal{F}}(X, A) \quad \subset \quad S_*(X, A)$$

Standard results in group theory imply that the last morphism on the top line is an isomorphism, and the preceding theorem shows that the last morphism on the second line is an isomorphism. Therefore if we pass to homology we obtain an isomorphism from $H_*(X - U, A - U)$ to $H_*(X, A)$, which is precisely the statement of the Excision Property.■

The same methods also yield the following result:

PROPOSITION 7. *If U and V are open subsets of a topological space, then the maps in singular homology induced by the inclusions $(U, U \cap V) \subset (U \cup V, V)$ are isomorphisms.*■

Axioms (E.1) and (E.2) follow immediately from the preceding discussion.

Mayer-Vietoris sequences

One of the most useful results for computing fundamental groups is the Seifert-van Kampen Theorem. There is a similar principle that can be applied to find the homology groups of a space X presented as the union of two open subsets U and V; in fact, the result in homology does not require any connectedness hypotheses on the intersection.

THEOREM 8. (Mayer-Vietoris Sequence for open sets in singular homology.) *Let X be a topological space, and let U and V be open subsets such that $X = U \cup V$. Denote the inclusions of U and V in X by i_U and i_v respectively, and denote the inclusions of $U \cap V$ in U and V by g_U and g_V respectively. Then there is a long exact sequence*

$$\cdots \to H_{q+1}(X) \to H_q(U \cap V) \to H_q(U) \oplus H_q(V) \to H_q(X) \to \cdots$$

in which the map from $H_*(U) \oplus H_*(V)$ to $H_*(X)$ is given on the summands by $(i_U)_*$ and $(i_V)_*$ respectively, and the map from $H_*(U \cap V)$ to $H_*(U) \oplus H_*(V)$ is given on the factors by $-(g_U)_*$ and $(g_V)_*$ respectively (note the signs!!).

Theorem 8 yields data type (T.2) and axiom (E.3) for singular homology.

Proof. Let $\mathcal{U}$ be the open covering of X whose sets are U and V, and let $S_*^{\mathcal{U}}(X)$ be the chain complex of all $\mathcal{U}$-small chains in $S_*(X)$. Then we have

$$S_*^{\mathcal{U}}(X) \quad = \quad S_*(U) \;+\; S_*(V) \quad \subset S_*(X)$$

(note that the sum is not direct) and hence we also have the following short exact sequence of chain complexes, in which the injection is given by the chain map whose coordinates are $-(g_U)_\#$ and $(g_V)_\#$ and the surjection is given on the respective summands by $(i_U)_\#$ and $(i_V)_\#$:

$$0 \longrightarrow S_*(U \cap V) \longrightarrow S_*(U) \oplus S_*(V) \longrightarrow S_*^{\mathcal{U}}(X) \longrightarrow 0$$

The Mayer-Vietoris sequence is the long exact homology sequence associated to this short exact sequence of chain complexes combined with the isomorphism $H_*^{\mathcal{U}}(X) \cong H_*(X)$.■

We have noted that one also has a Mayer-Vietoris sequences in simplicial homology, but for much different types of subspaces (in particular, the assumption is that a polyhedron is the union of two subcomplexes, and every subcomplex is closed and usually not open in P). Specifically, if $\mathbf{K}_1$ and $\mathbf{K}_2$ are subcomplexes of some $\mathbf{K}$, where $(P, \mathbf{K})$ is a simplicial complex, then the corresponding Mayer-Vietoris sequence has the following form:

$$\cdots \to H_{q+1}(\mathbf{K}) \to H_q(\mathbf{K}_1 \cap \mathbf{K}_2) \to H_q(\mathbf{K}_1) \oplus H_q(\mathbf{K}_2) \to H_q(\mathbf{K}) \to \cdots$$

It is possible to construct a unified framework that will include both of these exact sequences, but we shall not do so here because it involves numerous further results about simplicial complexes. However, it is important to note that in general one does NOT have a Mayer-Vietoris sequence in singular homology for presentations of a space X as a union of two closed subsets, and *this even fails for compact subsets of the* 2*-sphere.*

Example. Let $P \subset \mathbb{R}^2$ be the Polish circle constructed in `polishcircle.pdf` and `polishcircleA.pdf`, which is the union of the graph of $\sin(1/x)$ for $0 < |x| 1$ and the three closed line segments joining $(0, 1)$ to $(0, -2)$, $(0, -2)$ to $(1, -2)$, and $(1, -2)$ to $(1, \sin 1)$; there is a sketch of P in `polishcircleA.pdf`. By the discussion in the two references, P is a compact arcwise connected subset of the plane, and one can use the same argument as in Proposition 2 and Corollary 3 of `polishcircle.pdf` to prove that if K is compact and locally connected and $h : K \to P$ is continuous, then $h[K]$ lies in a contractible open subset of P and hence $H_q(P) = 0$ if $q \neq 0$ (by arcwise connectedness we have $H_0(\Gamma) \cong \mathbb{Z}$). Now let B be the set of points (x, y) in $\mathbb{R}^2$ satisfying

$$0 \quad \leq x \quad \leq 1 \quad \text{and } \textbf{either}$$

$$-2 \quad \leq \quad y \quad \leq \quad \sin(1/x) \quad \text{if} \quad x \neq 0 \quad \textbf{or} \quad -2 \quad \leq \quad y \quad \leq \quad 1 \quad \text{if} \quad x = 0 \; .$$

In the drawing on the first page of `polishcircleA.pdf`, B corresponds to the "closed bounded region whose boundary is P," and it follows immediately that $B = \text{Interior}(B) \cup P$, where the two subsets on the right hand side are disjoint, and that B is the closure of Interior(B). It is straightforward to show that the closed line segment $[0, 1] \times \{-\frac{3}{2}\}$ is a strong deformation retract of B; specifically, the retraction r sends (x, y) to $(x, -\frac{3}{2})$ and the homotopy is given by $t \cdot r(x, y) + (1 - t) \cdot (x, y)$. Therefore we know that the singular homology groups of both P and B are zero in all positive dimensions.

Viewing $\mathbb{R}^2 \subset S^2$ in the usual way, let $A = S^2 - \text{Interior}(B)$; then the observations in the preceding paragraph imply that $A \cap B = P$.

If there was an exact Mayer-Vietoris sequence in singular homology of the form

$$\cdots \to H_q(P) \to H_q(A) \oplus H_q(B) \to H_q(S^2) \to H_{q-1}(P) \cdots$$

then the results of the preceding paragraph would imply that $H_q(A) \cong H_q(S^2)$ for all $q \geq 2$, and in particular that the map $H_2(A) \to H_2(S^2)$ is nontrivial. Now A is a proper subset of S^2, and it is elementary to prove the following result:

LEMMA 9. *If $n > 0$ and A is a proper subset of S^n, then the inclusion map induces the trivial homomorphism from $H_n(A)$ to $H_n(S^n) \cong \mathbb{Z}$.*

Proof of Lemma 9. If $\mathbf{p}$ is a point of S^n that does not lie in A, then the homology map defined by inclusion factors as a composite

$$H_n(A) \to H_n(S^n - \{\mathbf{p}\}) \to H_n(S^n)$$

and this map is trivial because the complement of $\mathbf{p}$ is homeomorphic to $\mathbb{R}^n$ and the n-dimensional homology of the latter is trivial.■

This result and the discussion in the paragraphs preceding the lemma yield a contradiction; the source of this contradiction is our assumption that there is an exact Mayer-Vietoris sequence for $S^2 = A \cup B$, and therefore no such sequence can exist.

WHAT GOES WRONG IN THE EXAMPLE? In order to obtain an exact Mayer-Vietoris sequence for closed subsets, one generally needs an extra condition on the regularity of the inclusion maps. One standard type of condition on the closed subsets is that one can find arbitrarily small open neighborhoods such that the subsets are deformation retracts of these neighborhoods. This definitely fails for $P \subset \mathbb{R}^2$. In fact, one can use the methods of `polishcircle.pdf` and `polishcircleA.pdf` to show that P has a cofinal system of decreasing open neighborhoods $\{W_m\}$ such that $W_{m+1} \subset W_m$ is a homotopy equivalence for all m and each neighborhood is homotopy equivalent to S^1. Since $H_1(P) = 0$, there cannot be arbitrarily small open neighborhoods $V \supset P$ such that P is a deformation retract of V (if, say, $V \subset W_1$ and we choose n such that $W_n \subset V$, then the nontriviality of $H_1(W_n) \to H_1(W_1)$ implies the nontriviality of $H_1(W_n) \to H_1(V)$ and hence V cannot be contractible).

A more refined analysis yields axiom (E.4).

THEOREM 10. (Naturality of Mayer-Vietoris sequences) *In the setting of Theorem 5, assume we are given a map of triads f from $(X_1; U_1, V_1)$ to $(X_2; U_2, v_2)$. Then there for all integers q there is a commutative ladder as below in which the horizontal lines represent the long exact Mayer-Vietoris sequences of Theorem 5 and the vertical maps are all induced by f:*

$$\begin{array}{ccccccccccc} \cdots & \to & H_{q+1}(X_1) & \to & H_q(U_1 \cap V_1) & \to & H_q(U_1) \oplus H_q(V_1) & \to & H_q(X_1) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & H_{q+1}(X_2) & \to & H_q(U_2 \cap V_2) & \to & H_q(U_2) \oplus H_q(V_2) & \to & H_q(X_2) & \to & \cdots \end{array}$$

Proof. For $i = 1, 2$ let $\mathcal{F}(i)$ denote the open covering of X_i by U_i and V_i. Then we have the following commutative diagram of chain complexes whose rows are short exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & S_*(U_1) \cap S_*(V_1) & \to & S_*(U_1) \oplus S_*(V_1) & \to & S_*^{\mathcal{F}(1)}(X_1) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & S_*(U_2) \cap S_*(V_2) & \to & S_*(U_2) \oplus S_*(V_2) & \to & S_*^{\mathcal{F}(2)}(X_2) & \to & 0 \end{array}$$

The theorem follows by taking the long exact commutative ladder associated to this diagram.■

For the sake of completeness, we note that our work thus far yields the following conclusion, which corresponds to one of the axioms for a **simplicial** homology theory.

THEOREM 11. *Suppose that the pair (X, A) is obtained by regularly attaching a k-cell to A, and let $D \subset X$ denote the image $f[D^k]$, and let $S \subset X$ denote the image $f[S^{k-1}]$. Then the* *inclusion of* (D, S) *in* (X, A) *induces isomorphisms of singular homology groups from* $H_*(D, S)$ *to* $H_*(X, A)$.

Proof. In `algtop-notes.tex` this statement appeared as Theorem VII.6.1 and was derived as a consequence of axioms (A.1)–(A.5), (B.1)–(B.3), (C.1), (D.1)–(D.4) and (E.1)–(E.4). Since we have shown all of these hold for our construction of singular homology, the proof in the cited reference applies directly to yield the stated result.■

II.4 : Equivalence of simplicial and singular homology

(Hatcher, §§ 2.1 – 2.3)

We now have all the tools we need for verifying axiom (D.1), and as noted before this completes the justification of the applications in Unit VII of `algtop-notes.pdf`.

THEOREM 1. *Let* $(X, \mathbf{K})$ *be a simplicial complex, let* $(A, \mathbf{L})$ *determine a subcomplex, and let* $\theta_* : H_*(\mathbf{K}, \mathbf{L}) \to H_*(X, A)$ *be the natural transformation from simplicial to singular homology that was described previously. Then* θ_* *is an isomorphism.*

Proof. The idea is to apply Theorem I.1.8 on natural transformations of homology theories on simplicial complex pairs. In order to do this, we must check that singular homology for simplicial complexes satisfies the five properties $(a)-(e)$ listed shortly before the statement of I.1.8. Property (c), which gives the homology of a finite set, is verified in Proposition IV.1.4, and Properties (a), (b), (d) and (e) — which involve long exact sequences, the homology of a contractible space (more precisely, a simplex), excision for adjoining a single simplex, and the homology of a point — are respectively established in Theorem II.2.2, Corollary II.2.5, Theorem II.3.8, and the discussion following the problem stated after Corollary 1.1.4. Since all these properties hold, Theorem I.1.8 implies that the map θ_* must be an isomorphism for all simplicial complex pairs.■

II.5 : Polyhedral generation, direct limits and uniqueness

(Hatcher: 2.1–2.3, 3.F)

None of the material in this section will be used subsequently in these notes, so the reader may proceed directly to the next unit without loss of continuity. Since the material is optional, there will be less motivation, fewer details, and more reliance on references for topics not covered elsewhere in the course.

Here is one particularly important example to illustrate the preceding sentence: Theorem VI.8.1 in Eilenberg and Steenrod shows that *the restrictions of two singular homology theories are naturally isomorphic on the full subcategory of the underlying space pairs* (P, P') *for simplicial complex pairs* $(P', \mathbf{K}') \subset (P, \mathbf{K})$ (*i.e.*, the mappings are arbitrary continuous maps of pairs and not just subcomplex inclusions of one pair in another), and we shall use this fact without further discussion.

As indicated earlier, the key idea in extending simplicial to singular homology is approximating a space X by continuous maps of polyhedra into X, and axiom (C.3) is basically a formalization of this idea.

Polyhedral generation

This property, which is (C.3) on our list, is definitely less elementary than the ones we have discussed thus far, but for a number of reasons this seems to be the best place to verify it. One reason is that it only figures in proving the uniqueness of singular homology up to isomorphism (something that was never used in Unit VII of `algtop-notes.pdf`), and the reader may skip the rest of this section without loss of continuity.

In fact, we shall prove a modified version of (C.3); the reasons for making changes are given below.

THEOREM 1. (Polyhderal generation, slightly weakened) *If (X, A) is a pair of topological spaces, and let $u \in H_q(X, A)$, then there is a simplicial complex pair $(\mathbf{K}, \mathbf{K}')$ with $(P', \mathbf{K}') \subset (P, \mathbf{K})$ and a continuous map of pairs*

$$f : (P, P') \longrightarrow (X, A)$$

such that u is in the image of the map f_* from $H_q(P, P')$ to $H_q(X, A)$. Furthermore, if $(\mathbf{K}, \mathbf{K}')$ is a simplicial complex pair with underlying space pair (P, P') and $v \in H_q(P, P')$ maps trivially to $H_q(X, A)$ under the map f_*, then there is another simplicial complex pair $(Q', \mathbf{L}') \subset (Q, \mathbf{L})$ and continuous functions $h : (P, P') \to (Q, Q')$ and $g : (Q, Q') \to (X, A)$ such that the following hold:

(*i*) *The composite $g \circ h$ is homotopic to f.*

(*ii*) *We have $0 = h_*(v) \in H_q(Q, Q')$.*

This property has been well known to most (and perhaps nearly all) mathematicians who have worked extensively with algebraic topology (in particular, it is an immediate consequence of results on geometric realizations of semisimplicial sets; one reference suitable for a course at this level is J. P. May, *Simplicial Objects in Algebraic Topology*, University of Chicago Press, Chicago IL, 1982).

Note. This version of (C.3) is slightly weaker than the one stated in `algtop-notes.pdf`, in which the maps $(P, P') \to (Q, Q')$ were required to come from *subcomplex inclusions*. We have made this adjustment because the weaker statement is much easier to verify (for the original version, considerably more information involving simplicial complexes is needed) and the weakened version of (C.3) suffices for proving the uniqueness theorem that we want.

We shall use Hatcher's concept of Δ-complex explicitly in the course of the proof, and we shall also need a few properties of such objects.

LEMMA 2. *A finite Δ-complex in the sense of Hatcher is (compact and) Hausdorff.*

Although this property is dismissed as "obvious" on page 104 of Hatcher, some care seems appropriate because quotient spaces of compact Hausdorff spaces are not necessarily Hausdorff (of course they must be compact), so we shall outline the argument here. Hatcher's complex is constructed by taking a finite disjoint union of simplices and identifying selected subsets of faces with the same dimension. In abstract terms, this construction starts with a compact Hausdorff space X (which is the disjoint union of the simplices) and factors out an equivalence relation $\mathcal{R}$ whose graph in $X \times X$ is a closed subset of the latter (verify this explicitly!). One can then use point set topology to prove that the quotient space is Hausdorff. There is a particularly clear account of the proof in Theorem A.5.4 on page 252 of the following text (note that there are several texts on algebraic topology by the same author in the same series, so the precise title is particularly important here):

> **W. S. Massey.** *Algebraic Topology: An Introduction*, Graduate Texts in Mathematics Vol. 56, *Springer-Verlag, New York NY*, 1977.

Another important fact about Δ-complexes is that they are always homeomorphic to simplicial complexes. In fact, the second barycentric subdivision of the Δ-complex decomposition is always a simplicial decomposition, so one can actually say slightly more:

> *If* $(\mathbf{K}, \mathbf{K}')$ *is a* Δ*-complex pair with underlying space pair* (P, P')*, then the second barycentric subdivision induces a simplicial complex structure such that* P' *corresponds to a simplicial complex.*

This follows from Exercise 2.1.23 on page 133 of Hatcher, and a full proof is given in Theorem 16.41 on pages 148–149 of the following text:

> **B. Gray.** *Homotopy Theory: An Introduction to Algebraic Topology, Pure and Applied Mathematics Vol.* 64. *Academic Press, New York*, 1975.

It will be useful to introduce some notation for iterated faces of a simplex; specifically, if we are given a sequence $\mathbf{i} = (i_1, \cdots, i_r)$ such that $0 \leq i_t \leq q - t + 1$ for all t, the iterated face map $\partial_{\mathbf{i}} : \Delta_{q-r} \to \Delta_q$ will denote the composite of the ordinary face operators $\partial_{i_r} \circ \cdots \circ \partial_{i_1}$.

We now have enough machinery to prove the polyhderal generation property.

Proof of Theorem 1. By construction u is represented on the chain level by a singular chain $y = \sum_j n_j T_j$ such that the coefficients n_j are integers and the maps $T_j : \Delta_q \to X$ are continuous such that $dy \in S_{q-1}(A)$ (which is equivalent to saying that the image of y in $S_q(X, A)$ is a relative cycle). Form a Δ-complex P and a continuous map $g : P \to X$ by starting whose q-simplices σ_j are in 1–1 correspondence with the maps T_j, and identify two $(q - r)$-dimensional faces $\partial_{\mathbf{i}}\sigma \subset \sigma_k$ and $\partial_{\mathbf{j}}\beta \subset \sigma_m$ if $T_k|\partial_{\mathbf{i}}\Delta_q = T_m|\partial_{\mathbf{j}}\Delta_q$. Define g so that its restriction to σ_j is T_j for all j, and let $P' \subset P$ be the Δ-subcomplex of all $(q - 1)$-simplices that have nontrivial coefficients in the absolute chain dy, which by our choice automatically lifts back to $S_{q-1}(A)$. It follows immediately that g passes to a map of pairs, and by the preceding discussion there is a simplicial complex structure on P for which P' is a subcomplex (namely, the second barycentric subdivision). This completes the proof of the first part of the result.

Suppose now that $(\mathbf{K}, \mathbf{K}')$ is a simplicial complex pair with underlying space pair (P, P') and $v \in H_q(P, P')$ maps trivially to $H_q(X, A)$ under the map f_*. If ω is a linear ordering of the vertices of $\mathbf{K}$, then Theorems I.1.7 and II.4.1 imply that v is represented by a relative cycle $z = \sum_j n_j T_j$ in $C_q(\mathbf{K}^\omega)$, where the sum ranges over certain q-simplices of $\mathbf{K}$ that are not in $\mathbf{K}'$ such that $dz \in C_q\big((\mathbf{K}')\big)$. If this cycle goes to zero in $H_q(X, A)$, then there is a singular chain $c = \sum_j m_\ell V_\ell \in S_{q+1}(X)$ such that $dc = z + b$ for some $b \in S_q(A)$; the latter condition implies that b is a linear combination $\sum_j p_n W_n$ for singular simplices $W_n : \Delta_q \to A$. One can now construct a Δ-complex structure $\mathbf{M}$ as in the preceding discussion, and the condition $dc = z + b$ implies that $\mathbf{K}$ is a subcomplex of $\mathbf{M}$; if Q is the underlying space of $\mathbf{M}$, then $P \subset Q$ and the data defining c yield a continuous extension g of f. Let $\mathbf{M}'$ be the union of $\mathbf{K}'$ and the q-simplices in $\mathbf{M}$ corresponding to the singular simplices W_n, let $Q' \subset Q$ be the underlying space of $\mathbf{M}'$, and let $h : (P, P') \to (Q, Q')$ be the inclusion map. By construction $g[Q'] \subset A$ and the map of pairs $f : (P, P') \to (X, A)$ determines an extension to $g : (Q, Q') \to (X, A)$ such that $h_*(v) = 0$, and the final statement in the conclusion of the theorem follows immediately from this.■

We shall merely state what we need and use Chapter VIII of Eilenberg and Steenrod for proofs and other details whenever possible. We shall be working with *quasi-ordered sets* $(D, \leq)$ which satisfy the reflexive and transitive conditions for partial orderings (the symmetric condition $a \leq b$ and $b \leq a \Rightarrow a = b$ is dropped); as in the partially ordered case, a quasi-ordered set defines a category whose objects are the elements of the set, and if $d_1, d_2 \in D$ there is one morphism $d_1 \to d_2$ if $d_1 \leq d_2$ and there are no morphisms otherwise. A quasi-ordered sets $(D, \leq)$ is said to be **directed** if it satisfies the following condition:

($\star$) *If $x, y \in D$ then there is some $z \in P$ such that $x, y \leq z$.*

Linearly ordered sets and lattices are obvious examples for which this condition holds. We are particularly interested in the following special case and certain constructions involving it:

Example. Let $\mathbb{R}^\infty$ be the set of all infinite sequences of real numbers $(x_1, x_2, \cdots)$ such that all but finitely many x_k are zero, and consider the set of $\mathbb{P}$ all simplicial complexes $(P, \mathbf{K})$ in $\mathbb{R}^\infty$ such that the vertices of each simplex are unit vectors (a single nonzero coordinate, which is equal to 1); by finiteness each subspace P of this type lies in some $\mathbb{R}^N \subset \mathbb{R}^\infty$ given by all sequences such that $x_k = 0$ for $k > N$. Condition ($\star$) holds because the union of two such complexes contains both of them. Note that *every simplicial complex is isomorphic to a complex in* $\mathbb{P}$.

Definition. A *directed system* $\{B_x : x \in D\}$ in a category $\mathbb{A}$ is a covariant functor B from the category defined by $(D, \leq)$ to $\mathbb{A}$, and a morphism of directed systems in $\mathbb{A}$, from $\{B_x : x \in D\}$ to $\{B'_x : x \in D'\}$, is a natural transformation of functors.

The simplest way to motivate the concept of direct limit is to look a simple class of examples. Suppose that we have an increasing sequence $\{G_n\}$ of groups (*i.e.*, G_n is a subgroup of G_{n+1} for all positive integers n). Then it is fairly easy to form a limiting object G_∞ which is essentially a monotone union of the groups G_n. More generally, if may view an object L in $\mathbb{A}$ as a directed system $\{\bullet\}(L)$ defined on the category $\{\bullet\}$ with a single morphism (and a single object), then we may define direct limits as follows:

Definition. Given a directed system $B : D \to \mathbb{A}$, a natural transformation $\varphi : B \to \{\bullet\}(L)$ is a *direct limit* if it has the universal mapping property: If $\omega : B \to \{\bullet\}(M)$ is another natural transformation, then there is a unique morphism $h : L \to M$ such that $\{\bullet\}(h) \circ \varphi = \omega$.

The usual sort of argument yields the following standard uniqueness and functoriality results:

THEOREM 3. *(i) If $\varphi : B \to \{\bullet\}(L)$ and $\omega : B \to \{\bullet\}(M)$ are direct limits, then there is a unique isomorphism $h : L \to M$ such that $\{\bullet\}(h) \circ \varphi = \omega$.*

(ii) If $\varphi : B \to \{\bullet\}(L)$ and $\omega : C \to \{\bullet\}(M)$ are direct limits with values in the same category and $F : B \to C$ is a map of directed systems, then there is a unique map of direct limit objects $f_\infty : L \to M$ such that $\{\bullet\}(f_\infty) \circ \varphi = \omega \circ F$. Furthermore, the construction sending F to f_∞ is (covariantly) functorial.■

The usefulness of the direct limit concept obviously depends upon a reasonable existence statement for such objects. The following can be found in Eilenberg and Steenrod:

THEOREM 4. *If $\mathbb{A}$ is a category of groups with operators (for example, modules over a ring or the category of groups), then every directed system in $\mathbb{A}$ has a direct limit $\varphi : B \to \{\bullet\}(L)$, and it has the following properties:*

(i) Every element of L has the form $\varphi_x(u)$ for some $x \in D$ and some $u \in B_x$.

(ii) if $v \in B_x$ is such that $\varphi_x(v)$ is the trivial element, then there is some $y \geq x$ such that v maps to the trivial element of D_y.■

A (the) direct limit of φ is often denoted by dir lim (B) or similar notation, and the universal map is often denoted by something like dir lim (ω).

There are some clear analogies between the conclusions in Theorems 1 and 4; needless to say, we are going to exploit these similarities.

An isomorphism theorem for singular homology theories

Following Eilenberg and Steenrod, we shall say that a pair of compact Hausdorff spaces (P, P') is $\mathbb{P}-$*triangulable* (it can be triangulated) if it is the underlying space pair for a simplicial complex pair $(\mathbf{K}, \mathbf{K}')$ in $\mathbb{P}$. The restriction to $\mathbb{P}$ is added to obtain a family of bounded cardinality which is large enough to include all isomorphism types of simplicial complex pairs.

Let (X, A) be a pair of topological spaces, and define a directed system $\mathbb{P}(X, A)$ whose elements are given by a $\mathbb{P}-$triangulable pair (P, P') and a continuous map of pairs $f : (P, P') \to (X, A)$. The quasi-ordering on such objects

$$f : (P, P') \to (X, A) \qquad \leq \qquad g : (Q, Q') \to (X, A)$$

(often shortened to $f \leq g$) is given by the existence of a continuous mapping of pairs $h : (P, P') \to (Q, Q')$ such that $f \simeq g \circ h$. The following result shows that $\mathbb{P}(X, A)$ satisfies the required condition $(\star)$ for a directed system:

LEMMA 5. *The quasi-ordered set $\mathbb{P}(X, A)$ is directed.*

Proof. We begin with a general observation about $\mathbb{P}(X, A)$. Suppose that (P_1, P_1') is a pair of subcomplexes in $\mathbb{P}$ which is simplicially isomorphic to (P, P'), and let $J : (P_1, P_1') \to (P, P')$ be a simplicial isomorphism. Then we have $f \leq f \circ J \leq f$ in $\mathbb{P}(X, A)$.

Suppose now that we are given $f : (P, P') \to (X, A)$ and $g : (Q, Q') \to (X, A)$. Clearly we can construct a subcomplex pair (P_1, P_1') which is isomorphic to (P, P') and disjoint from Q (take the vertices of P_1 to be vertices which are not in Q). Let J be a simplicial isomorphism as in the preceding paragraph, and take the map α of pairs from the disjoint union $(P_1, P_1') \amalg (Q, Q') = (P_1 \amalg Q,\ P_1' \amalg Q')$ to (X, A) whose restriction to (P_1, P_1') is $f \circ J$ and whose restriction to (Q, Q') is g. By construction we have $\alpha \geq f \circ J, g$, and since $f \circ J \geq f$ we also have $\alpha \geq f, g$, which is exactly what we needed to prove.■

THEOREM 6. *There is a canonical isomorphism Γ from the direct limit of $\{H_*(P_\alpha, P_\alpha') : \alpha \in \mathbb{P}\}$ to $H_*(X, A)$, and it is natural with respect to continuous maps of pairs.*

Proof. For each nonnegative integer q there is a natural transformation

$$\gamma_q : \{H_q(P_\alpha, P_\alpha') : \alpha \in \mathbb{P}\} \longrightarrow \{\bullet\}\big(H_q(X, A)\big)$$

defined by the homology homomorphisms associated to the continuous mappings $g_\alpha : (P_\alpha, P_\alpha') \to (X, A)$, and by the universal mapping property these yield homomorphisms

$$\text{dir lim}\,(\omega) : \text{dir lim}\,\{H_q(P_\alpha, P_\alpha') : \alpha \in \mathbb{P}\} \longrightarrow H_q(X, A)\ .$$

Theorems 1 and 4 combine to imply that these homomorphisms are isomorphisms.

Suppose now that we are given a continuous map of pairs $\varphi : (X, A) \to (Y, B)$. Composition with φ defines a map of directed sets $\mathbb{P}(\varphi) : \mathbb{P}(X, A) \to \mathbb{P}(Y, B)$, and this construction is functorial with respect to continuous maps of pairs. If we apply the homology functor H_*, we obtain a map of the corresponding direct systems of abelian groups, and by Theorem 3 we obtain a natural transformation from dir lim $\{H_q(P_\alpha, P'_\alpha) : \alpha \in \mathbb{P}\} to H_q(X, A)$; in this setting naturality is with respect to continuous maps of pairs. By Theorems 1 and 4 this natural mapping is an isomorphism.■

We now have the machinery we need to prove the following uniqueness theorem:

THEOREM 7. *Suppose that (h_*, ∂) and (h'_*, ∂') satisfy the following weak versions of the axioms for a singular homology theory:*

(a) *All the data types except possibly* (T.2) *and* (T.4), *and all the axioms except possibly* (A.6), (D.5), (E.3)–(E.4) *and* (C.3).

(b) *The weaker version of* (C.3) *corresponding to Theorem* 1.

Then there is a unique natural isomorphism $\lambda : h_* \to h'_*$ such that for each point p the isomorphism $\lambda_{\{p\}}$ commutes with the normalization isomorphisms $h_0(\{p\}) \cong \mathbb{Z}$ and $h'_0(\{p\}) \cong \mathbb{Z}$.

Proof. The proof of Theorem 6 is valid for an arbitrary axiomatic singular homology satisfying the conditions in the conclusion of Theorem 1, so the conclusion of Theorem 6 remains valid for an abstract singular homology theory satisfying the hypotheses in the present theorem. In other words, if $k = h$ or h' then $k(X, A)$ is naturally isomorphic to the direct limit of the system $\{k_*(P_\alpha, P'_\alpha)$, where $\alpha \in \mathbb{P}(X, A)$.

For each pair of spaces (X, A) the previously cited uniqueness theorem in Eilenberg and Steenrod yields a natural isomorphism λ of directed systems

$$h_*(P_\alpha, P'_\alpha)_{\alpha \in \mathbb{P}(X,A)} \longrightarrow h'_*(P_\alpha, P'_\alpha)_{t \in \mathbb{P}(X,A)} .$$

The direct limits of these systems are $h_*(X, A)$ and $h'_*(X, A)$ respectively, and therefore one obtains a direct limit isomorphism λ_∞ from $h_*(X, A)$ to $h'_*(X, A)$.

The naturality of this isomorphism follows from Theorem 3.■

III. Additional geometric applications

We shall begin this unit by constructing the data and verifying the axioms relating our construction of singular homology to the fundamental group. The remaining sections deal with some additional standard applications of homology to questions involving roots of equations (Section 2), fixed points and integer invariants of spaces which can be used to distinguish homotopy types in certain cases (Section 4), a topological definition for the dimension of a space (Section 5), and questions about the extent to which certain line integrals over closed curves are path-independent (Section ∞).

These applications are just a few simple examples of what can be done with homology groups, and the following examples show that homology theory can be a very powerful tool in studying questions about homotopy classes of maps from one space to another. Both can be found in Hatcher.

HOPF'S THEOREM. *Let P be a finite connected n-dimensional polyhedron, where $n \geq 1$. Then there is an abelian group structure on the set of homotopy classes $[P, S^n]$ such that the torsion free part is isomorphic to the set of homomorphisms* $\mathrm{Hom}\,(\,H_n(P),\, H_n(S^n) \cong \mathbb{Z})$ *and the torsion subgroup is isomorphic to the torsion subgroup of $H_{n-1}(P)$.*

Since the result obviously also holds if P is merely homeomorphic to a polyhedron, it follows that two continuous maps from S^n to itself are homotopic if and only if they induce the same homomorphism from $H_n(S^n) \cong \mathbb{Z}$ to itself; such a homomorphism is determined by its value on a generator and thus determines a number called the *degree*. We shall look at this concept further in Section 2.

SIMPLY CONNECTED CASE OF J. H. C. WHITEHEAD'S THEOREM. *Suppose that P and Q are finite* simply connected *polyhedra and $f : P \to Q$ is a continuous map such that for each $i \geq 0$ the induced map of homology $f_* : H_i(P) \to H_i(Q)$ is an isomorphism. Then f is a homotopy equivalence.*

The converse is an immediate consequence of the functoriality and homotopy invariance of homology groups. There are versions of Whitehead's Theorem for connected finite polyhedra that are not simply connected, but once again we do not have the background needed to formulate such a result here. However, it is important to note that the non-simply connected case requires stronger hypotheses than the condition that f defines isomorphisms of ordinary homology groups (specifically, one needs to know that f induces an isomorphism of fundamental groups and isomorphisms on the homology groups of the universal covering spaces for P and Q).

III.1 : Homology and the fundamental group

(Hatcher, §§ 2.A, 3.G)

Axiom (D.5) formulates a simple but important relationship between the fundamental group $\pi_1(X, x)$ of a pointed arcwise connected space and the homology group $H_1(X) \cong H_1(X, \{x\})$.

Definition. Let $[S^1] \in H_1(S^1)$ be the homology class represented by the singular 1-simplex

$$T(1-s, s) \;=\; \big(\cos 2\pi s,\, \sin 2\pi s\,\big)$$

so that T corresponds to the standard counterclockwise parametrization of the unit circle under the identification of $[0,1]$ with the 1-simplex whose vertices are $(1,0)$ and $(0,1)$. The *Hurewicz* (hoo-RAY-vich) map $h : \pi_1(X,x) \to H_1(X)$ is given by taking a representative f of $\alpha \in \pi_1(X,x)$ and setting $h(\alpha) = f_*([S^1])$. By homotopy invariance, this class does not depend upon the choice of a representative, and it is natural with respect to basepoint preserving continuous maps.

PROPOSITION 1. *The Hurewicz map h is a group homomorphism.*

Proof. The discussion on pages 166–167 of Hatcher provides a good conceptual summary of the proof. For the sake of completeness we shall add a few details.

Let $\mathfrak{p} : \Delta_2 \to [0,1]$ be the map sending $(t_0,t_1,t_2) \in \Delta_2$ to $t_2 + \frac{1}{2}t_1 \in [0.1]$. Geometrically, $\mathfrak{p}$ is the composite of the perpendicular projection from Δ_2 onto the edge $\mathbf{e}_0\mathbf{e}_2$ followed by the linear homeomorphism from the latter to $[0,1]$ sending $\mathbf{e}_0$ to 0 and $\mathbf{e}_2$ to 1. Represent $u, v \in \pi_1(X,x)$ by $f, g : [0,1] \to X$, and let $c : [0,1] \to X$ be the concatenation $f + g$. If α is the linear homeomorphism from Δ_1 to $[0,1]$ sending vertex $\mathbf{e}_t$ to t (where $t = 0,1$), then direct calculation yields the identities

$$\partial_2 \circ \mathfrak{p} \circ h \quad = \quad f \circ \alpha, \quad \partial_0 \circ \mathfrak{p} \circ h \quad = \quad g \circ \alpha, \quad \partial_1 \circ \mathfrak{p} \circ h \quad = \quad c \circ \alpha$$

(compare the drawing on page 166 of Hatcher) so that we have

$$d_2(\mathfrak{p} \circ h) \quad = \quad f \circ \alpha + g \circ \alpha - c \circ \alpha \in S_1(X, \{x\}) \ .$$

By construction, the images of the three summands on the right hand side of this equation are $h(u)$, $h(v)$ and $-h(uv)$ respectively, and since the left hand side is a boundary it follows that $h(u) + h(v) - h(uv) = 0$, which is what we wanted to prove.∎

The preceding discussion and the theorem below show that the standard construction for singular homology has extra data type (T.2) and satisfies axioms (A.6) and (D.4); by the uniqueness result in the preceding unit, the same conclusions are true for an arbitrary axiomatic singular homology theory.

THEOREM 2. *If X is arcwise connected, then h is onto and its kernel is the commutator subgroup of $\pi_1(X,x)$.*

The assertion in the first sentence of the theorem is verified on page 167 of Hatcher; the proof of the assertion in the second sentence will take the remainder of this section.

Suppose that (X,x) is a pointed space such that X is arcwise connected. The *Eilenberg subcomplex* $\overline{S_*}(X) \subset S_*(X)$ is the chain subcomplex generated by all singular simplices $T : \Delta_q \to X$ which send each vertex of Δ_q to the chosen basepoint x.

PROPOSITION 3. *Under the conditions given above, the inclusion of the Eilenberg subcomplex defines an isomorphism in singular homology.*

Sketch of proof. For each $y \in X$ there is a continuous curve joining y to x, and hence for each singular 0-simplex given by a point y there is a singular 1-simplex $P(y)$ such that $P(y) \circ \partial_1$ is the constant function with value x and $P(y) \circ \partial_0$ is the constant function with value y; clearly it is possible to choose $P(x)$ to be the constant function, and we shall do so. Starting from this, we claim by induction on q that for each singular q-simplex $T : \Delta_q \to X$ there is a continuous map

$$P(T) : \Delta_q \times [0,1] \ \longrightarrow \ X$$

with the following properties:

(i) The restriction of $P(T)$ to $\Delta_q \times \{0\}$ is given by T, and the restriction of $P(T)$ to $\Delta_q \times \{1\}$ is given by a singular simplex in the Eilenberg subcomplex.

(ii) If T lies in the Eilenberg subcomplex, then $P(T)$ is equal to $T \times \mathrm{id}_{[0,1]}$.

(iii) For each face map $\partial_i : \Delta_{q-1} \to \Delta_q$ we have $P(T \circ \partial_i) = P(T) \circ (\partial_i \times \mathrm{id}_{[0,1]})$.

To complete the inductive step, one uses (iii) and the first property in (i) to define $P(T)$ on $\Delta_q \times \{0\} \ \cup \ \partial\Delta_q \times [0,1]$, and then one extends this to all of $\Delta_q \times [0,1]$ using the Homotopy Extension Property.

Let i denote the inclusion of the Eilenberg subcomplex, and define a map ρ from $S_*(X)$ to the Eilenberg subcomplex by taking $\rho(T)$ to be the restriction of $P(T)$ to $\Delta_q \times \{1\}$. The property (iii) ensures that ρ is a chain map, and we also know that $\rho \circ i$ is the identity on the Eilenberg subcomplex. The proof of the proposition will be complete if we can show that $i \circ \rho$ is chain homotopic to the identity. The proof of this is very similar to the proof of homotopy invariance. Let $\mathbf{P}_{q+1} \in S_{q+1}(\delta_q \times [0,1])$ be the standard chain used in that proof, and define

$$E(T) \quad = \quad \big(P(T) \big)_{\#} \mathbf{P}_{q+1} \ .$$

Then the properties of $\mathbf{P}_{q+1}$ and its boundary imply this defines a chain homotopy from the identity to $i \circ \rho$.■

Conclusion of the proof of Theorem 2. We shall use the following commutative diagram:

$$\begin{array}{ccccc}
F_2(X,x) & \xrightarrow{\ \mathbf{abel}\ } & \overline{S_2}\,(X) & \xrightarrow{\ =\ } & \overline{S_2}\,(X) \\
\downarrow \delta & & \downarrow d_2 & & \downarrow d_2 \\
F_1(X,x) & \xrightarrow{\ \mathbf{abel}\ } & \overline{S_1}\,(X) & \xrightarrow{\ =\ } & \overline{S_1}\,(X) \\
\downarrow \mathbf{can} & & \downarrow \mathbf{can}' & & \downarrow \mathbf{class} \\
\pi_1(X,x) & \xrightarrow{\ \mathbf{abel}\ } & \pi_1^{\mathbf{ab}}(X,x) & \xrightarrow{\ h'\ } & H_1(X)
\end{array}$$

Many items in this diagram need to be explained. On the bottom line, $\pi_1^{\mathbf{ab}}$ denotes the abelianization of the fundamental group formed by factoring out the (normal) commutator subgroup, and the Hurewicz map has a unique factorization as $h' \circ \mathbf{abel}$, where **abel** refers to the canonical surjection from π_1 to its quotient modulo the commutator subgroup. The groups $F_j(X,x)$ are the free groups on the free generators for the Eilenberg subcomplexes $\overline{S_*}\,(X)$, and **abel** generically denotes the passage from free groups to the corresponding free abelian groups. The maps d_2 and class are merely the relevant maps for the Eilenberg subcomplex, the map **can** ′" is the abelianization of the map **can** taking a free generator $T : \Delta_1 \to X$, which is merely a closed curve in X based at x, to its homotopy class in the fundamental group. Finally, δ is a nonabelian boundary map defined on free generators by

$$\delta(T) \quad = \quad [T \circ \partial_2] \cdot [T \circ \partial_0] \cdot [T \circ \partial_1]^{-1} \ .$$

Observe that the composite $\mathbf{can} \circ \delta$ is trivial and hence its abelianization $\mathbf{can}' \circ d_2$ is also trivial.

Proof that the Hurewicz map is onto. Suppose we are given a cycle $z = \sum_i \ n_i T_i$ in the Eilenberg subcomplex. and we let $\gamma(T_i) \in F_1(X,x)$ denote the free generator corresponding to T_i. Then it follows immediately from the commutative diagram that the homology class u represented by z satisfies

$$u \ = h(\alpha) \ , \qquad \text{where} \quad \alpha \ = \ \prod_i \big[\mathbf{can}\big(\gamma(T_i)\big) \big]^{n_i} \ .$$

Proof that the reduced Hurewicz map (i.e., its factorization through the abelianization of the fundamental group) is injective. Suppose that $h(\alpha) = 0$ and that the free generator $y \in F_1(X, x)$ represents α. Then it follows that $\mathbf{abel}(y) = d_2(w)$ for some 2-chain w, and if $w' \in F_2(X, x)$ projects to w then $y = \delta(w) \cdot v$, where v lies in the commutator subgroup of $F_1(X, x)$. Since $\mathbf{can} \circ \delta$ is trivial, it follows that the image of y in $\pi_1^{\mathbf{abel}}$ is trivial. Finally, since the image of y in π_1 is α, it also follows that the image of α in $\pi_1^{\mathbf{abel}}$ is trivial, or equivalently that α lies in the commutator subgroup.■

The results of this section and the normalization axioms for singular homology theories imply a strong converse to the Seifert-van Kampen Theorem for describing the fundamental group of a space X which is the union of arcwise connected open subset U and V. Namely, if the images of $\pi_1(U)$ and $\pi_1(V)$ generate $\pi_1(X)$, then the intersection is arcwise connected.

PROPOSITION 4. *Suppose that X is a topological space which is the union of arcwise connected open subsets U and V (such that the base point lies in $U \cap V$), and assume that $U \cap V$ is* not *arcwise connected, and let $\Gamma \subset \pi_1(X)$ be the subgroup generated by the images of $\pi_1(U)$ and $\pi_1(V)$. Then Γ has infinite index in $\pi_1(X)$.*

Since one of the simplest examples for Theorem 3 is the circle expressed as a union of two open arcs whose intersections are two small closed arcs, the conclusion of Theorem 3 is obvious in this special case and thus the theorem shows that something similar happens in every other example.

Proof. By Theorem 1, it will suffice to show that the image of Γ in $H_1(X)$ has infinite index in the latter group, for if a subgroup $K \subset G$ has finite index, then its image in the abelianization $G/[G, G]$ will also have finite index (verify this; it is an elementary exercise in group theory$^{(\star)}$).

Theorem 1 implies that the image of Γ in $H_1(X)$ is equal to the image of the inclusion induced homomorphism

$$H_1(U) \;\oplus\; H_1(V) \quad\longrightarrow\quad H_1(X)$$

in the Mayer-Vietoris exact sequence associated to the decomposition $X = U \cup V$:

$$H_1(U) \oplus H_1(V) \to\to H_1(X) \to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(X) \cong \mathbb{Z} \to 0$$

Since 0-dimensional homology groups are free abelian on their sets of arc components, this sequence is given more concretely as follows, in which Π denotes the set of arc components of $U \cap V$, the maps from $\mathbb{Z}^\Pi$ to the two $\mathbb{Z}$ factors are given up to sign by adding coordinates, and the map from $\mathbb{Z} \oplus \mathbb{Z}$ is also addition:

$$H_1(U) \oplus H_1(V) \to H_1(X) \to \mathbb{Z}^\Pi \to \mathbb{Z} \oplus \mathbb{Z} \to H_0(X) \cong \mathbb{Z} \to 0$$

Since we are assuming that Π contains at least two elements, it follows that the map $\mathbb{Z}^\Pi \to \mathbb{Z} \oplus \mathbb{Z}$ has a nontrivial kernel and hence by exactness the map $H_1(X) \to \mathbb{Z}^\Pi$ has an infinite image. One more application of exactness implies that the image of the map $H_1(U) \oplus H_1(V) \to H_1(X)$ must have infinite index, and by the remarks at the beginning of this paragraph the same is true for the image of the subgroup $\Gamma \subset \pi_1(X)$. As noted in the first paragraph of the proof, this means that Γ must have infinite index in the fundamental group of X.■

III.2 : Degree theory

(Hatcher, § 2.2)

Definition. If $n > 0$ and $f : S^n \to S^n$ is a continuous mapping, then the *degree* of f is the unique integer d such that the map $f_* : H_n(S^n) \to H_n(S^n)$ is multiplication by d (recall that $H_n(S^n) \cong \mathbf{Z}$ and every homomorphism of the latter to itself is multiplication by some integer).

Several properties of the degree are immediate:

(1) If f is the identity, then the degree of f is 1.

(2) If f is a constant map, then the degree of f is 0.

(3) If f and g are homotopic, then their degrees are equal.

(4) If f and g are continuous maps from S^n to itself, then the degree of $f \circ g$ is equal to the degree of f times the degree of g.

(5) If h is a homeomorphism of S^n to itself, then the degree of h and h^{-1} is ± 1, and the degree of $h \circ f \circ h^{-1}$ is equal to the degree of f.

(6) If $n = 1$ and $f(z) = z^m$ (complex arithmetic), then the degree of f is equal to m.

The last property is the only one which is nontrivial. It follows because (a) the map f_* from $\pi_1(S^1, 1) \cong \mathbf{Z}$ is multiplication by m, (b) the Hurewicz map from $\pi_1(S^1, 1)$ to $H_1(S^1)$ is an isomorphism, (c) the Hurewicz map defines a natural transformation of functors from the fundamental group to 1-dimensional singular homology.■

For all $n \geq 2$, there is a standard recursive process for constructing continuous maps from S^n to itself with arbitrary degree.

PROPOSITION 1. *Let $f : S^{n-1} \to S^{n-1}$ be a continuous mapping of degree d, and let $\Sigma(f) : S^n \to S^n$ be defined on $(x, t) \in S^n \subset \mathbb{R}^n \times \mathbb{R}$ by*

$$\Sigma(f)\big(x, t\big) \quad = \quad \left(\sqrt{1 - t^2} f(x),\, t\right) .$$

Then the degree of $\Sigma(f)$ is also equal to d.

COROLLARY 2. *If $n \geq 1$ and d is an arbitrary integer, then there exists a continuous mapping $g : S^n \to S^n$ whose degree is equal to d.*

The case $n = 1$ of the corollary is just (6), above, and the proposition supplies the inductive step to show that if the corollary is true for $(n-1)$ then it is also true for n.■

Proof of Proposition 1. We should check first that the map $\Sigma(f)$ is continuous. This is immediate from the formula for all points except the north and south poles, and at the latter one can check directly that if $\varepsilon > 0$ then we can take $\delta = \varepsilon$.

Define D^n_+ and D^n_- to be the subsets of S^n on which the last coordinates are nonnegative and nonpositive respectively. It follows immediately that S^n is formed from S^{n-1} by attaching two n-cells corresponding to $D^n_\pm$. This and the vanishing of the homology of disks in positive dimensions imply that all the arrows in the diagram below are isomorphisms:

$$H_{*-1}(S^{n-1}) \leftarrow H_*(D^n_+, S^{n-1}) \to H_*(S^n, D^n_-) \leftarrow H_*(S^n)$$

Furthermore, the mappings f and $\Sigma(f)$ determine homomorphisms from each of these homology groups to themselves such that the following diagram commutes:

$$\begin{array}{ccccccc}
H_{n-1}(S^{n-1}) & \xleftarrow{\cong} & H_n(D^n_+, S^{n-1}) & \xrightarrow{\cong} & H_n(S^n, D^n_-) & \xleftarrow{\cong} & H_*(S^n) \\
\downarrow f_* & & \downarrow \Sigma(f)_* & & \downarrow \Sigma(f)_* & & \downarrow \Sigma(f)_* \\
H_{n-1}(S^{n-1}) & \xleftarrow{\cong} & H_n(D^n_+, S^{n-1}) & \xrightarrow{\cong} & H_n(S^n, D^n_-) & \xleftarrow{\cong} & H_*(S^n)
\end{array}$$

It follows immediately that the degrees of f and $\Sigma(f)$ must be equal.■

Here is another basic property:

PROPOSITION 3. *If $f : S^n \to S^n$ is continuous and the degree of f is nonzero, then f is onto.*

Proof. If the image of f does not include some point $\mathbf{p}$, then f_* has a factorization of the form

$$H_n(S^n) \ \to\ H_n(S^n - \{\mathbf{p}\}) \ \to\ H_n(S^n)$$

and this homomorphism is trivial because the middle group is zero.■

Linear algebra and degree theory

We shall start with orthogonal transformations.

PROPOSITION 4. *Suppose that T is an orthogonal linear transformation of $\mathbb{R}^n$, where $n \geq 2$, and let $f_T : S^{n-1} \to S^{n-1}$ be the corresponding homeomorphism of S^{n-1}. Then the degree of f_T is equal to the determinant of T.*

Sketch of proof. We shall use a basic fact about orthogonal matrices; namely, if A is an orthogonal matrix then there is another orthogonal matrix B such that $B \cdot A \cdot B^{-1}$ is equal to a block sum of 2×2 rotation matrices plus a block sum of 1×1 matrices such that at most one of the latter has an entry of -1 (and the rest must have entries of 1).

Every 2×2 rotation matrix can be joined to the identity by a path consisting entirely of 2×2 rotation matrices. Therefore it follows that f_T is homotopic to f_S, where S is a diagonal matrix with at most one entry equal to -1 and all others equal to 1. Clearly the degrees of f_S and f_T are equal, and likewise the determinants of S and T must be equal (by continuity of the determinant and the fact that its value for an orthogonal matrix is always ± 1). Thus the proof reduces to showing that the degree of f_S is equal to -1 if there is a negative diagonal entry and is equal to 1 if there are no negative diagonal entries. — In fact, the second statement is obvious since T and f_T are identity mappings in this case.

Therefore everything reduces to showing that the degree of f_S is equal to -1. We can use Proposition 2 to show that the result is true for all n if it is true for $n = 2$, and the truth of the result when $n = 2$ follows immediately from Property (6) of degrees that was stated at the beginning of this document.■

We shall now consider an arbitrary invertible linear transformation T from $\mathbb{R}^n$ to itself. Such a map is a homeomorphism and thus extends to a map $T^\bullet$ of one point compactifications from S^n to itself.

THEOREM 5. *In the setting above, the degree of $T^\bullet$ is equal to the sign of the determinant of T.*

The proof of this result requires some additional input.

LEMMA 6. *Suppose that we are given a continuous curve T_t defined for $t \in [0,1]$ and taking values in the set of all invertible linear transformations on $\mathbb{R}^n$ (equivalently, invertible $n \times n$ matrices). Then $T_0^\bullet$ is homotopic to $T_1^\bullet$.*

Proof of Lemma 6. We would like to define a homotopy by the formula $H_t = T_t^\bullet$, and we can do so if and only if the latter is continuous at every point of $\{\infty\} \times [0,1]$. The latter in turn reduces to showing the following: *For each $t \in [0,1]$ and $M > 0$ there are numbers $\delta > 0$ and $P > 0$ such that $|s - t| < \delta$ and $|v| \geq P$ imply $|T_s(v)| \geq M$.*

Let $\|T\|$ be the usual norm of a linear transformation given by the maximum value of $|T|$ on the unit sphere. It follows immediately that the norm is a continuous function in (the matrix entries associated to) T. It follows that

$$|T_s(v)| \quad \geq \quad \|T_s^{-1}\| \cdot |v|$$

and since the inverse operation is also continuous it follows that $\|T_s^{-1}\|$ is a continuous function of s. In particular, if $\|T_t^{-1}\| = B > 0$ then we can find $\delta > 0$ such that $|s - t| < \delta$ implies $\|T_s^{-1}\| > B/2$, and hence if $|v| > 2M/B$ and $|t - s| < \delta$ then $T_s(v)| \geq M$, as required.■

Proof of Theorem 5. Both the degree of $T^\bullet$ and the sign of the determinant are homomorphisms from invertible matrices to $\{\pm 1\}$, and therefore it will suffice to prove the theorem for a set of linear transformations which generate all the invertible linear transformations. Not surprisingly, we shall take this set to be the linear transformations given by the elementary matrices.

Let $E_{i,j}$ denote the $n \times n$ matrix which has a 1 in the (i,j) entry and zeros elsewhere. Then the function sending $t \in [0,1]$ to $I + tE_{i,j}$ defines a curve from the elementary matrix $I + E_{i,j}$ to the identity. Therefore the associated linear transformation determines a map which is homotopic to the identity, and consequently the degree and determinant sign agree for elementary linear transformations given by adding a multiple of one row to another.

Similarly, if $D(k,r)$ is a diagonal matrix which has ones except in the k^{th} position and a positive real number r in the latter position, then there is a continuous straight line curve joining the matrix in question to the identity, and this matrix takes values in the group of invertible diagonal matrices. It follows that the degree and determinant sign agree for elementary linear transformations given by multiplying one row by a positive constant.

We are now left with elementary matrices given by either multiplying one row by -1 or by interchanging two rows. These two types of matrices are similar, so both the degrees and determinant signs are equal in each case. Therefore it will suffice to check that the degree and determinant sign agree when one considers an elementary matrix given by multiplying a single row by -1.

By Proposition 2 and the invariance of our numerical invariants under similarity, it will suffice to consider the case where $n = 2$ and we are multiplying the second row by -1. Let $W \subset \mathbb{R}^2$ be the open disk of radius 2 about the origin, so that there is a canonical homeomorphism from $W - \{\mathbf{0}\}$ to $S^1 \times (0,2)$. Now the map $T^\bullet$ sends $S^2 - \{\mathbf{0}\}$ to itself and likewise for W and S^1. Excision and homotopy invariance now yield the following chain of isomorphic homology groups:

$$H_1(S^1) \leftarrow H_1(W - \{\mathbf{0}\}) \rightarrow H_2(W, W - \{\mathbf{0}\}) \leftarrow H_2(S^2, S^2 - \{\mathbf{0}\}) \longrightarrow H_2(S^2)$$

As in Proposition 3, one has associated maps of homology groups to form a corresponding commutative diagram, and from this diagram one sees that the degree of $T^\bullet$ is equal to the degree of the map determined by $T^\bullet$ on S^1. Since the map on S^1 is merely the mapping sending z to z^{-1}, it follows that the degree is equal to -1, and of course this is the same as the sign of the determinant.∎

The Fundamental Theorem of Algebra

One can use degree theory to prove the Fundamental Theorem of Algebra. All proofs of the latter involve some analysis and plane topology, and one advantage of the degree-theoretic proof is that the role of topology is particularly easy to recognize. This proof can also be generalized to obtain a generalization of the Fundamental Theorem of Algebra to polynomials with quaternionic coefficients (this was done by Eilenberg and Niven in the nineteen forties).

We start with an argument that is similar to the proof in the last part of Theorem 5.

PROPOSITION 7. *The map ψ^m of the complex plane sending z to z^m (where m is a positive integer) extends continuously to a map of one point compactifications sending the point at infinity to itself, and the degree of the compactified map is equal to m.*

Proof. The existence of a continuous extension follows because if $M > 0$ then $|z| > M^{1/m}$ implies $|z^m| > M$.

It follows that ψ^m sends $\mathbb{C} - \{\mathbf{0}\}$ to itself. Of course, the map also sends S^1 to itself and this map has degree m, so a diagram chase plus the naturality of the Hurewicz homomorphism imply that ψ^m_* is multiplication by m on $H_1(\mathbb{C} - \{\mathbf{0}\}) \cong \mathbf{Z}$. Diagram chases now show that ψ_* is multiplication by m on

$$H_2(\mathbb{C}, \mathbb{C} - \{\mathbf{0}\}) \quad \cong \quad H_2(S^2, S^2 - \{\mathbf{0}\}) \quad \cong \quad H_2(S^2)$$

and thus the degree of the compactified map is equal to m.∎

The following result is standard.

PROPOSITION 8. *If p is a nonconstant monic polynomial, then p extends continuously to a map of one point compactifications sending the point at infinity to itself.*

Sketch of proof. We need to show that if $M > 0$ then there is some $\rho > 0$ such that $|z| > \rho$ implies $|p(z)| > M$. One easy way of doing this is to begin by writing p as follows:

$$p(z) \quad = \quad z^m \cdot \left(1 \; + \; \frac{a_{n-1}}{z} \; + \; \cdots \; + \; \frac{a_0}{z^n}\right)$$

If we write the expression inside the parentheses as $1 + b(z)$, then it is clear that if $|z|$ is sufficiently large (say $|z| > N$) then $|b(z)| < \frac{1}{2}$. It follows immediately that if $M > 0$ and $|z| > 2M^{1/m} + N$ then $|p(z)| > M$.∎

The Fundamental Theorem of Algebra will now be a consequence of Proposition 3 and the following generalization of Proposition 8:

PROPOSITION 9. *If p is a nonconstant monic polynomial of degree $m \geq 1$, then the degree of the compactified map $p^\bullet$ is equal to m.*

Proof. It will suffice to show that $p^\bullet$ is homotopic to $(\psi^m)^\bullet$.

Define a homotopy from ψ^m to p on the set where $|z| \geq N + 1$ by $h_t(z) = z^m(1 + t\,b(z)\,)$. By the Tietze Extension Theorem, one can extend this to a homotopy over all of $\mathbb{C}$. As in the previous argument, if $M > 0$ and $|z| > 2M^{1/m} + N + 1$ then $|h_t(z)| > M$ for all t. One can then argue as in the first paragraph of the proof of Lemma 6 to show that $p^\bullet$ is homotopic to $(\psi^m)^\bullet$.∎

Local degree theory

In many situations it is necessary or useful to have a version of degree theory which applies to certain continuous maps from an open subset of $\mathbb{R}^n$ into $\mathbb{R}^n$. There is an elegant and definitive treatment of this subject in Section IV.5 of the following text:

> **A. Dold**. Lectures on Algebraic Topology (Second Edition), Grundlehren der mathematischen Wissenschaften, Bd. 200. Springer-Verlag, New York *etc.*, 1980. ISBN: 3540586601.

INFINITE-DIMENSIONAL LERAY-SCHAUDER DEGREE THEORY. A basic theme in linear functional analysis is the partial generalization of finite-dimensional linear algebra to some infinite-dimensional settings, and one reason for doing this is to develop techniques for studying solutions to linear differential and integral equations. In the 1930s J. Leray and J. Schauder constructed an infinite-dimensional version of degree theory, with a corresponding motivation to analyze certain nonlinear differential and integral equations. This theory is described in Part II of the book by Brown cited below; the article by Mawhin summarizes the history of the theory along with its generalizations and applications.

> **R. Akerkar.** Nonlinear functional analysis. Narosa Publishing House, New Delhi, 1999.
>
> **R. F. Brown.** A Topological Introduction to Nonlinear Analysis (2^{nd} Ed.). Birkhäuser, Boston, 2004.
>
> **J. Mawhin.** Leray-Schauder degree: a half century of extensions and applications. Topological Methods in Nonlinear Analysis **14** (1999), 195–228.

`http://www.mat.univie.ac.at/~gerald/ftp/book-nlfa/nlfa.pdf`

III.3 : Simplicial approximation

(Hatcher, § 2.C)

The treatment in Hatcher is fairly standard, so we shall only discuss a few issues here$^{(\star)}$.

PROPOSITION 1. *Let $g : \mathbf{K} \to \mathbf{L}$ be a simplicial map, let $|g|$ be the associated continuous map of underlying topological spaces, and let λ_* denote the standard natural transformation obtained from the chain complex inclusion $C_*(\mathbf{K}) \to S_*(P)$, where P is the polyhedron with simplicial decomposition $\mathbf{K}$. Then $\lambda_* \circ g_* = |g|_* \circ \lambda_*$.*

This follows immediately from the construction of λ, for if $\mathbf{v}_0 \ \cdots \ \mathbf{v}_q$ is one of the free generators for $C_q(\mathbf{K})$, then its image under the associated simplicial chain map associated to g is $g(\mathbf{v}_0) \ \cdots g(\mathbf{v}_q)$i, and under the chain map $\lambda(\mathbf{L})_\#$ this goes to $|g|_\# \circ \lambda(\mathbf{K})_\# \, (\mathbf{v}_0 \ \cdots \ \mathbf{v}_q)$.■

COROLLARY 2. *Suppose that $(P, \mathbf{K})$ and $(Q, \mathbf{L})$ are simplicial complexes, and let $f : P \to Q$ be continuous. Suppose that $r > 0$ and $g : B^r(\mathbf{K}) \to \mathbf{L}$ are such that g is a simplicial approximation to f, and let $\beta_r : C_*(\mathbf{K}) \to C_*(B^r(\mathbf{K})\,)$ be the iterated barycentric subdivision map. Then $f_* \circ \lambda_* = \lambda_* \circ g_* \circ (\beta_r)_*$.*

Sketch of proof. We have an analog of β_r defined from $S_*(P)$ to itself, and by the results leading to the proof of the Excision Property this map is chain homotopic to the identity. From this it follows that $|g|_* \circ \lambda_* = \lambda_* \circ g_* \circ (\beta_r)_*$. Since g is a simplicial approximation to f we know that $f_* = |g|_*$, and if we make this substitution into the equation in the preceding sentence we obtain the assertion in the corollary.■

Of course, the point of the corollary is that one can compute the map in homology associated to f using the simplicial approximation g.

Given a continuous function f as above, one natural question about simplicial approximations is to find the value(s) of r for which there is a simplicial approximation $g : B^r(\mathbf{K}) \to \mathbf{L}$. The result below shows that in many cases we must take r to be very large.

PROPOSITION 3. *Suppose that $(P, \mathbf{K})$ and $(Q, \mathbf{L})$ are simplicial complexes, and let $f : P \to Q$ be continuous. Let $r_0(f) > 0$ be the smallest value of r such that f is homotopic to a simplicial map $g : B^r(\mathbf{K}) \to \mathbf{L}$. Then the following hold:*

(i) The number $r_0(f)$ depends only upon the homotopy class of f.

(ii) If the set of homotopy classes $[P, Q]$ is infinite, then for each positive integer M there are infinitely many homotopy classes $[f_n]$ such that $r_0(f_n) > M$.

Proof. The first part follows immediately from the definition, so we turn out attention to the second. Recall that a simplicial map is completely determined by its values on the vertices of the domain.

Suppose now that $\mathbf{L}$ has b vertices and $B^r(\mathbf{K})$ has a_r. There are b^{a_r} different ways of mapping the vertices of $B^r(\mathbf{K})$ to those of $\mathbf{L}$; although some of these might not arise from a simplicial map, we can still use this to obtain a finite upper bound on the number of simplicial maps from $B^r(\mathbf{K})$ to $\mathbf{L}$, and we also have a finite upper bound on the number of simplicial maps from $B^r(\mathbf{K})$ to $\mathbf{L}$ for all $r \leq M$ if M is any fixed positive integer. It follows that there are only finitely many homotopy classes for which $r_0 \leq M$.■

In particular, by the results of Section V.1 we can apply this proposition to $[P, Q]$ where P and Q are both homeomorphic to S^n for some $n \geq 1$.

III.4 : The Lefschetz Fixed Point Theorem

(Hatcher, § 2.C)

Once again the treatment in Hatcher is fairly standard, so we shall only concentrate on a few issues.

The Euler characteristic

In `algtop-notes.pdf` we discussed the Euler characteristic of a regular cell complex; our purpose here is prove extensions of the main results on Euler characteristics to finite cell complexes as defined in Section I.3 of these notes, and the crucial result is Theorem I.3.9, which shows that the singular homology of a cell complex is isomorphic to the homology of a cellular chain complex whose q-dimensional group may be viewed as a free abelian group on the set of q-cells.

Notation. Let (C, d) be a chain complex over the rationals such that only finitely many chain groups C_q are nonzero and the nonzero groups are all finite-dimensional vector spaces over the rationals.

(*i*) Set c_q equal to the dimension of C_q.

(*ii*) Set b_q equal to the rank of d_q.

(*iii*) Set z_q equal to the dimension of the kernel of d_q.

(*iv*) Set h_q equal to the dimension of $H_q(C)$.

It follows immediately that these numbers are defined for all q and are equal to zero for all but finitely many a.

The equation involving the numbers of faces for a convex linear cell depends upon the following algebraic result.

PROPOSITION 1. *In the setting above we have*

$$\sum_q (-1)^q c_q \quad = \quad \sum_q (-1)^q h_q \ .$$

Proof. The main idea of the argument is given on pages 146 – 147 of Hatcher. In analogy with the discussion there, we have $c_q - z_q = b_q$ and $z_q - b_{q+1} = h_q$, so that

$$\sum_q (-1)^q h_q \quad = \quad \sum_q (-1)^q (z_q - b_{q+1}) \quad = \quad \sum_q (-1)^q z_q \ - \ \sum_q (-1)^q b_{q+1} \quad =$$

$$\sum_r (-1)^r z_r \ + \ \sum_r (-1)^r b_r \quad = \quad \sum_q (-1)^q c_q$$

proving that the two sums in the proposition are equal.■

COROLLARY 2. *Suppose that $(X, \mathcal{E})$ is a finite cell complex with c_q cells in dimension $q \geq 0$, and suppose that $H_q(X)$ is isomorphic to a direct sum of β_q infinite cyclic groups plus a finite group. Then we have*

$$\sum_{q \geq 0} (-1)^q c_q \quad = \quad \sum_{q \geq 0} (-1)^q \beta_q \ .$$

The statement regarding convex linear cells follows immediately from Corollary 11 and Proposition 5. — In general, the topologically invariant number on the right hand side is called the **Euler characteristic** of X and is written $\chi(X)$.

Proof. Let A_* be the chain complex over the rational numbers with $A_q = C_q(X, \mathcal{E})_{(0)}$ and the differential given by rationalizing d_q. It then follows that $\dim A_q = c_q$ and $\dim H_q(A) = \beta_q$. The corollary then follows by applying Proposition 1.■

The Lefschetz number

From the viewpoint of these notes, the Lefschetz number is obtained using the traces of various maps on rational chain groups or cohomology groups. The proof that the alternating sum of traces is the same for simplicial chains and simplicial homology is a special case of the following result:

PROPOSITION 3. *Suppose that C_* is a chain complex of rational vector spaces such that each C_q is finite-dimensional and only finitely many are nontrivial, and let $T : C_* \to C_*$ be a chain map. Then*

$$\sum_q (-1)^q \operatorname{trace} T_q \quad = \quad \sum_q (-1)^q \operatorname{trace} (T_*)_q \ .$$

The proof of this combines the method of Proposition 1 with the following result:

LEMMA 4. *Let V be a finite-dimensional vector space over a field, let W be a vector subspace, and suppose that $T : V \to V$ is a linear transformation such that $T[W] \subset W$. Let T_W be the associated linear transformation from W to itself, and let $T_{V/W}$ denote the linear transformation from V/W to itself which sends $\mathbf{v} + W$ to $T(\mathbf{v}) + W$ for all $\mathbf{v} \in V$ (this is well-defined). Then* $\mathrm{trace}\,(T) = \mathrm{trace}\,(T_W) + \mathrm{trace}\,(T_{V/W})$.

Proof of Lemma 4. Pick a basis $\mathbf{w_1}, \cdots, \mathbf{w}_k$ for W and extend it to a basis for V by adding vectors $\mathbf{u}_{k+1}, \cdots, \mathbf{u}_n$. It follows that the vectors $\mathbf{u}_{k+1} + W, \cdots, \mathbf{u}_n + W$ form a basis for V/W. If we now let $\mathbf{v}$ denote either $\mathbf{v}$ or $\mathbf{w}$ and as usual write

$$T(\mathbf{v}_j) \quad = \quad \sum_i a_{i,j}\, \mathbf{v}_i$$

then the traces of T, T_W and $T_{V/W}$ are given by the sums of the scalars $a_{i,i}$ from 1 to n in the case of T, from 1 to k in the case of T_W, and from $k+1$ to n in the case of $T_{V/W}$.■

As noted above, Proposition 3 follows by applying the same method used in Proposition 1 with the dimensions c_q, z_q, b_q and h_q replaced by the traces of the corresponding linear transformations.■

Vector fields on S^2

We may think of a *tangent vector field* on the sphere S^2 as a continuous map $\mathbf{X} : S^2 \to \mathbb{R}^3$ such that $\mathbf{X}(\mathbf{u})$ is perpendicular to $\mathbf{u}$ for all $\mathbf{u} \in S^2$ (in other words, the value of $\mathbf{X}$ at a point $\mathbf{u}$ in S^2 is the tangent vector to a curve passing through $\mathbf{u}$). One can use the Lefschetz Fixed Point Theorem to prove the following fundamental result on such vector fields.

THEOREM 5. *If $\mathbf{X}$ is a tangent vector field on S^2, then there is some $\mathbf{u} \in S^2$ such that* $\mathbf{X}(\mathbf{u}) = \mathbf{0}$.

Proof. Suppose that the vector field is everywhere nonzero. If we set

$$\mathbf{Y}(\mathbf{u}) \quad = \quad |\mathbf{X}(\mathbf{u})|^{-1} \cdot \mathbf{X}(\mathbf{u})$$

then $\mathbf{Y}$ is a continuous vector field such that $|\mathbf{Y}|$ is always equal to 1, so that $\mathbf{Y}$ defines a continuous map from S^2 to itself. By the perpendicularity condition we know that $\mathbf{Y}(\mathbf{u}) \neq \mathbf{u}$ for all $\mathbf{u}$, and therefore by the Lefschetz Fixed Point Theorem we know that the Lefschetz number of $\mathbf{Y}$ must be zero.

We now claim that $\mathbf{Y}$ defines a continuous map from S^2 to itself which is homotopic to the identity. Specifically, take the homotopy

$$H(\mathbf{u}, t) \quad = \quad \cos\left(\frac{t\pi}{2}\right) \cdot \mathbf{Y}(\mathbf{u}) \; + \; \sin\left(\frac{t\pi}{2}\right) \cdot \mathbf{u}$$

which which moves $\mathbf{u}$ to $\mathbf{Y}(\mathbf{u})$ along a 90° great circle arc. Since $\mathbf{Y}$ is homotopic to the identity, it follows that its Lefschetz number equals the Lefschetz number of the identity, which is $\chi(S^2) = 2$. This contradicts the conclusion of the preceding paragraph; the source of this contradiction was our assumption that $\mathbf{X}(\mathbf{u}) \neq \mathbf{0}$ for all $\mathbf{u}$, and therefore it follows that there is some $\mathbf{u}_0 \in S^2$ such that $\mathbf{X}(\mathbf{u}_0) = \mathbf{0}$.■

In fact, *the same argument goes through virtually unchanged for all even-dimensional spheres*. On the other hand, every odd-dimensional sphere does admit a tangent vector field which is everywhere nonzero. One quick way to construct an example is to take the vector field on $S^{2n+1} \subset \mathbb{R}^{2n+2}$ given by the formula

$$\mathbf{X}(x_1, x_2, x_3, x_4, \cdots, x_{2n+1}, x_{2n+2}) = (-x_2, x_1, -x_4, x_3, \cdots, -x_{2n+2}, x_{2n+1}) ;$$

if we view $\mathbb{R}^{2n+2}$ as $\mathbb{C}^{n+1}$, then the vector field sends a vector $\mathbf{z} = (z_1, \cdots, z_{n+1})$ to $i\,\mathbf{z}$.

Geometric interpretation of the Lefschetz number. Suppose that P is a polyhedron which is homeomorphic to a compact smooth manifold M (without boundary), and let $f : M \to M$ be a smooth self-map. Basic results on approximating mappings on smooth manifolds imply that f is homotopic to a smooth map $g : M \to M$ such that g has only finitely many fixed points and for each fixed point $x \in M$ the associated linear map of the tangent space $T(x)$ at x

$$L_f(x) = \mathbf{T}(g)_x : T(x) \longrightarrow T(x)$$

has the property that $L_f(x) - \mathrm{id}_{T(x)}$ is an isomorphism (in such cases the fixed point set is said to be isolated and nondegenerate). For each fixed point x one can define a *local fixed point index* $\Lambda(g)_x$ to be the sign of the determinant of $L_f(x) - \mathrm{id}_{T(x)}$. Under these conditions the Lefschetz number of g turns out to be given by

$$\Lambda(g) = \sum_{g(x)=x} \Lambda(g)_x .$$

Proving this is beyond the scope of these notes and requires the notion of *local fixed point index*. In the paper cited below, a set of axioms for fixed point indices of smooth maps is given, and Chapter 7 of the text by Dold explains how such indices are related to the Lefschetz number as described here:

> **M. Furi, M. P. Pera, and M. Spadini.** *On the uniqueness of the fixed point index on differentiable manifolds*. Fixed point theory and its applications **2004**, 251–259.

Generalizations of the Brouwer Fixed Point Theorem. One can view the Brouwer Fixed Point Theorem as a special case of the Lefschetz Fixed Point Theorem in which the polyhedron P is homeomorphic to a disk or simplex. More generally, we have the following:

THEOREM 6. *Suppose that P is a connected polyhedron such that $H_i(P, \mathbb{Q}) = 0$ for all $i > 0$, and let $f : P \to P$ be a continuous mapping. Then the Lefschetz number of f is equal to 1 and hence f has a fixed point.*

Proof. Since P is connected it follows that f induces the identity on $H_0(P; \mathbb{Q}) \cong \mathbb{Q}$, and since all higher dimensional rational homology groups vanish it follows that the Lefschetz number must be 1. The conclusion regarding fixed points now follows from the Lefschetz Fixed Point Theorem.■

A very similar argument yields another generalization in a somewhat different direction.

THEOREM 7. *Suppose that P is a connected polyhedron, and let $f : P \to P$ be a nullhomotopic continuous mapping. Then the Lefschetz number of f is equal to 1 and hence f has a fixed point.*

Proof. Since P is connected it follows that f induces the identity on $H_0(P; \mathbb{Q}) \cong \mathbb{Q}$, and since f is nullhomotopic all self maps of higher dimensional (rational) homology groups are trivial, so that the Lefschetz number of f must be 1. The conclusion regarding fixed points now follows from the Lefschetz Fixed Point Theorem.■

Finally, we should also mention an infinite-dimensional generalization of the Brouwer Fixed Point Theorem.

THEOREM 8. (Schauder Fixed Point Theorem) *Let C be a closed convex subset of the Banach space X, and suppose that $f : C \to C$ is a continuous self-map Which is also compact (i.e., the image of a bounded subset in C has compact closure). Then f has a fixed point in C.*■

Here is an online reference for a proof of this result:

`http://www.math.unl.edu/~s-bbockel1/933-notes/node5.html`

III.5 : Dimension theory

(Munkres, § 50)

In this section, we are interested in the following basic question:

> *Is there some purely topological way to describe the intuitive notion of $n-$dimensionality, at least for spaces that are relatively well-behaved?*

Of course, in linear algebra there is the standard notion of dimension, and this concept has far-reaching consequences for understanding dimensions in geometry. A topological approach to describing the dimensions of at least some spaces is implicit in our proof for Invariance of Dimension (see Proposition IV.2.16), which can be used to define a notion of dimension for topological spaces which locally look like an open subset of $\mathbb{R}^n$ for some fixed $n \geq 0$. There is an extensive literature on topological approaches to defining the dimensions of spaces. Our purpose here is to discuss one particularly important example known as the *Lebesgue covering dimension*; for reasonably well-behaved classes of spaces this is equivalent to other frequently used concepts of dimension. Here are some printed and online references for topological dimension theory:

> **W. Hurewicz and H. Wallman.** *Dimension Theory* (Revised Edition, Princeton Mathematical Series, Vol. 4). *Princeton University Press, Princeton*, 1996.
>
> **K. Nagami.** *Dimension Theory* (with an appendix by Y. Kodama, Pure and Applied Mathematics Series, Vol. 37). *Academic Press, New York*, 1970.
>
> **J. Nagata.** *Modern Dimension Theory* (Second Edition, revised and extended; Sigma Series in Pure Mathematics, Vol. 2). *Heldermann-Verlag, Berlin*, 1983.

`http://en.wikipedia.org/wiki/Lebesgue_covering_dimension`

`http://en.wikipedia.org/wiki/Dimension`

`http://en.wikipedia.org/wiki/Inductive_dimension`

FRACTAL DIMENSIONS. There are several notions of *fractal dimension* for subsets of $\mathbb{R}^n$ which depend on the way in which an object is embedded in $\mathbb{R}^n$ and not just the subset's underlying topological structure; for example, various standard examples of nonrectifiable curves in the plane have fractal dimensions which are numbers strictly between 1 and 2. Such objects are interesting for a variety of reasons, but they are beyond the scope of this course so we shall only give two online references here:

`http://en.wikipedia.org/wiki/Fractal_dimension`

The basic setting

We shall base our discussion upon the material in Section 50 of Munkres. For the sake of clarity we shall state the main definition and mention some standard conventions.

Definition. Let X be a topological space, let n be a nonnegative integer, and let $\mathcal{U}$ be an indexed open covering of X. Then we shall say that *the open covering* $\mathcal{U}$ *has order at most* n provided every intersection of the form

$$U_{\alpha(0)} \cap \cdots \cap U_{\alpha(n)}$$

is empty, and we shall say that *the space* X *has Lebesgue covering dimension* $\leq n$ provided every open covering $\mathcal{U}$ of X has a refinement $\mathcal{V}$ of order $\leq n$. Frequently we shall write $\dim X \leq n$ if the Lebesgue covering dimension is at most n.

We shall say that $\dim X = n$ (the Lebesgue covering dimension is equal to n) if $\dim X \leq n$ is true but $\dim X \leq n-1$ is not. By convention, the Lebesgue covering dimension of the empty set is taken to be -1, and we shall write $\dim X = \infty$ if $\dim X \leq n$ is false for all n.

Munkres states and proves many fundamental results about the Lebesgue covering dimension, and we shall not try to copy or rework most of his results here. Instead, our emphasis in this section will be on the following key issues:

(1) Describing precise connections between the topological theory of dimension as in Munkres and the algebraic notions of k-dimensional homology groups for various choices of k.

(2) Using the methods of these notes to give an alternate proof of Theorem 50.6 in Munkres; namely, if $A \subset \mathbb{R}^n$ is compact, then the topological dimension of A satisfies $\dim A \leq n$.

(3) Using algebraic topology to prove that the topological dimension of an n-dimensional polyhedron is in fact **equal** to n (the results in Munkres show that this dimension is at most n).

We shall begin by addressing the dimension question in (2); one reason for doing this is that the approach taken here will play a crucial role in our treatment of the subject.

MUNKRES, THEOREM 50.6. *If* A *is a compact subset of* $\mathbb{R}^n$*, then* $\dim A \leq n$.

Alternate proof. We know that there is some very large hypercube K of the form $[-M, M]^n$ which contains A, and we also know that A is closed in this hypercube. By Theorem 50.1 on pages 306–307 of Munkres, it is enough to show that the hypercube has dimension at most n. Since every hypercube has a simplicial decomposition with simplices of dimension $\leq n$, it will suffice to prove the following result:

LEMMA 1. *If* $P \subset \mathbb{R}^m$ *is a polyhedron with an* n*-dimensional simplicial decomposition, then the topological dimension of* P *is at most* n.

If we know this, then we know that the hypercube, and hence A, must have topological dimension $\leq n$.■

Proof of Lemma 1. Let $\mathcal{U}$ be an open covering of the hypercube K, and let $\varepsilon > 0$ be a Lebesgue number for $\mathcal{U}$. Using barycentric subdivisions, we can find an n-dimensional simplicial decomposition of K whose simplices all have diameter less than $\varepsilon/2$. Therefore if $\mathbf{v}$ is a vertex of this simplicial decomposition, then the open set $\mathbf{Openstar}(\mathbf{v})$ is contained in some element of $\mathcal{U}$. Now these sets form an open covering of K (see Section 2.C of Hatcher), and therefore these open stars form a finite open refinement of $\mathcal{U}$. Since an intersection of open stars $\cap_i$ **Openstar**($\mathbf{v}_i$) is nonempty if and only if the vertices $\mathbf{v}_i$ lie on a simplex from the underlying simplicial decomposition, the n-dimensionality of the decomposition implies that every intersection of $(n+2)$ distinct open stars must be empty. This is exactly the criterion for the covering by open stars to have order at most $(n+1)$. Therefore we have shown that $\mathcal{U}$ has a finite open refinement with at most this order, which means that the topological dimension of K is at most n.■

The discussions of the first and third issues are closely related, and they use the material on partitions of unity on pags 225–226 of Munkres (see Theorem 36.1 in particular).

Definitions. Let X be a $\mathbf{T_4}$ space, and let $\mathcal{U}$ be a finite open covering of X. Set $\mathbf{Vec}(\mathcal{U})$ equal to the (finite-dimensional) real vector space with basis given by the sets in $\mathcal{U}$, and define the **nerve** of $\mathcal{U}$, written $\mathfrak{N}(\mathcal{U})$, to be the simplicial complex whose simplices are given by all vertex sets of the form $U_{\alpha(0)}, \cdots, U_{\alpha(q)}$ such that

$$U_{\alpha(0)} \cap \cdots \cap U_{\alpha(q)} \neq \emptyset .$$

By construction, the vertices of this simplicial complex are all symbols of the form $[U_\alpha]$, where U_α is nonempty and belongs to $\mathcal{U}$.

If $\{ \varphi_\alpha \}$ is a partition of unity which is subordinate to (= dominated by) $\mathcal{U}$, then there is a *canonical map* k_φ from X to $\mathfrak{N}(\mathcal{U})$ given by the partition of unity:

$$k_\varphi(x) = \sum \varphi_\alpha(x) \cdot [U_\alpha]$$

Different partitions of unity yield different maps, but we have the following:

CLAIM: *For each finite open covering $\mathcal{U}$, all canonical maps from X to $\mathfrak{N}(\mathcal{U})$ are homotopic to each other.*

Proof of the claim. For each choice of x and canonical maps φ_0, φ_1, we know that the points $\varphi_i(x)$ lie on the simplex whose vertices are all $[U_\alpha]$ such that $x \in U_\alpha$. Thus the straight line segment joining $\varphi_0(x)$ to $\varphi_1(x)$ also lies on this simplex, and hence also lies in the nerve of $\mathcal{U}$. In other words, the image of the straight line homotopy from φ_0 to φ_1 is always contained in $\mathfrak{N}(\mathcal{U})$, and therefore the two canonical maps into $\mathfrak{N}(\mathcal{U})$ are homotopic.■

In the special case where $(P, \mathbf{K})$ is a simplicial complex and $\mathcal{U}$ is the open covering given by open stars of vertices (see Hatcher for the definitions), the canonical map(s) from P to the nerve of $\mathcal{U}$ can be described very simply as follows:

PROPOSITION 2. *Let P, $\mathbf{K}$ and $\mathcal{U}$ be as above, and for each vertex $\mathbf{v}$ of $\mathbf{K}$ define the* extended barycentric coordinate function $\mathbf{v}^* : P \to [0,1]$ *as follows: If $\mathbf{x} \in A$ for some simplex A which contains $\mathbf{v}$ as a vertex, let $\mathbf{v}^*(\mathbf{x})$ denote the barycentric coordinate of $\mathbf{x}$ with respect to $\mathbf{v}$, and if $\mathbf{x}$ lies on a simplex A which does not contain $\mathbf{v}$ as a vertex, set $\mathbf{v}^*(\mathbf{x}) = 0$ (it follows immediately that this map is well-defined and continuous). Define a map $\kappa : P \to \mathfrak{N}(\mathcal{U})$ by $\kappa(\mathbf{x}) = \sum \mathbf{v}^*(\mathbf{x}) \cdot \mathbf{v}$. Then κ defines a homeomorphism from P to $\mathfrak{N}(\mathcal{U})$, and every canonical map with respect to the open covering $\mathcal{U}$ is homotopic to κ.*

Sketch of proof. First of all, the barycentric coordinate functions are well-defined, for if $\mathbf{x}$ lies on a simplex A with vertex $\mathbf{v}$ and also on a simplex B for which $\mathbf{v}$ is not a vertex, then it follows that the barycentric coordinate of $\mathbf{x}$ with respect to $\mathbf{v}$ must be zero. The assertion that κ defines a homeomorphism from P to the nerve of $\mathcal{U}$ follows because κ maps the simplices of $\mathbf{K}$ bijectively to the simplices of $\mathfrak{N}(\mathcal{U})$; more precisely, there is a 1–1 correspondence of simplices and each simplex of $\mathbf{K}$ is sent to a simplex of the nerve by a bijective affine map.

Finally, the proof that κ is homeomorphic to a canonical map associated to a partition of unity follows from the same considerations which appear in the proof that two canonical maps are homotopic (for every $\mathbf{x} \in P$, there is a simplex in the nerve containing both $\kappa(\mathbf{x})$ and the value of a canonical map at $\mathbf{x}$).∎

Čech homology groups

The idea behind singular homology groups is that one approximates a space by maps **from** simplicial complexes (in particular, simplices) into a space X. Dually, the idea behind Čech homology groups is that one approximates a space by maps **into** simplicial complexes. Constructions of this type play an important role in the theory and applications of machinery from algebraic topology, but we shall only focus on what we need. As is often the case, the first step is to construct some necessary algebraic machinery.

Inverse systems and inverse limits

The definition of Čech homology requires the notion of *inverse limit*; special cases of this concept appear in Hatcher, but since we need the general case we must begin from scratch.

Definition. A *codirected set* is a pair $(A, \prec)$ consisting of a set A and a binary operation $\prec$ such that the following hold:

(a) (Reflexive Property) For all $x \in A$ we have $x \prec x$.

(b) (Transitive Property) If $x, y, z \in A$ are such that $x \prec y$ and $y \prec z$, then $x \prec z$.

(c) (Lower Bound Property) For all $x, y \in A$ there is some $w \in A$ such that $w \prec x$ and $w \prec y$.

These are similar to the defining conditions for a partially ordered set, but we do **not** assume the symmetric property (so $x \prec y$ and $y \prec x$ does not necessarily imply $x = y$), and the Lower Bound Property does not necessarily hold for a partially ordered set which is not linearly ordered. On the other hand, if a partially ordered set is a **lattice** (*i.e.*, finite subsets always have least upper bounds and greatest lower bounds), then it is a codirected set.

The basic example of a codirected set in Hatcher is given by the positive integers $\mathbb{N}^+$ with the **reverse** of the usual partial ordering, so that $a \prec b$ if and only if $b \geq a$.

Given a codirected set $(A, \prec)$, there is an associated category $\mathsf{CAT}\,(A, \prec)$ for which $\mathsf{Morph}\,(x, y)$ is nonempty if and only if $x \prec y$, and in this case $\mathsf{Morph}\,(x, y)$ contains exactly one element.

Definition. Let $(A, \prec)$ be a codirected set, and let $\mathbf{C}$ be a category. An *inverse system* in $\mathbf{C}$ indexed by $(A, \prec)$ is a covariant functor F from $\mathsf{CAT}\,(A, \prec)$ to $\mathbf{C}$. If $a \prec b$, then the value of F on the unique morphism $a \to b$ is frequently denoted by notation like $f_{a,b}$; in other words, $f_{a,b} = F(a \prec b)$.

There is a closely related concept of *inverse limit* for inverse systems. One can do this in purely categorical terms, but we are only interested in working with inverse limits over categories of modules. For inverse systems $F = \{F(a)\}$ of modules, the inverse limit

$$\lim_{\leftarrow} \quad = \quad \text{inv} \lim_{A} \ F(a) \quad = \quad \text{proj} \lim_{A} \ F(a)$$

is defined to be the set of all $x = (x_a)$ in $\prod_A \ F(a)$ such that for each $a \prec b$ we have $f_{a,b}(x_a) = x_b$. For each $a \in A$ the map p_a denotes projection onto the a-coordinate.

Inverse limits have the following universal mapping property, which in fact characterizes the construction.

PROPOSITION 3. *Suppose that F is an inverse system as above, and suppose that we are given a module L with maps $q_a : L \to F(a)$ such that $f_{a,b} \circ q_a = q_b$ whenever $a \prec b$. Then there is a unique homomorphism $h : L \to \lim_{\leftarrow} F(a)$ such that $g_a = f_a \circ h$ for all a.*

This is an immediate consequence of the definitions.■

There are straightforward analogs of the inverse limit construction for may categories (sets, compact Hausdorff spaces, groups, ...), and we shall leave the details of setting up such objects to the reader as an exercise.

Frequently it is important to recognize that inverse limits of directed systems can be given by inverse limits over "good" subobjects. We shall say that $B \subset A$ is a *codirected subobject* if B is a subset, the binary relation is the restriction of the binary relation on A, and the Lower Bound Property still holds on B (however, if $w \in A$ is such that $w \prec b, a$ we do not necessarily assume that $w \in B$; we only assume that there is some $w' \in B$ with $w' \prec a, b$). We shall say that such a object is *cofinal* if for each $x \in A$ there is some $y \in B$ such that $y \prec x$.

Example. Let γ be a cardinal number, and let $\mathsf{Cov}_{\gamma}(X)$ be the family of indexed open coverings of X such that the cardinality of the indexing set is at most γ. We shall say that an indexed open covering $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ is an *indexed refinement* of $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ if there is a map of indexing spaces $j : B \to A$ such that $V_\beta \subset U_{j(\beta)}$ for all β; note that if $\mathcal{V}$ is a refinement of $\mathcal{U}$ in the usual sense then by the Axiom of Choice we can always find a function j with the required properties. — Suppose now that X is a compact metric space and $\mathsf{FinCov}(X)$ is a set of all finite indexed open coverings whose indexing sets are subsets of the set $\mathbb{N}$ of nonnegative integers. If $\mathbb{A}$ is a subset of $\mathsf{FinCov}(X)$ such that for each $k > 0$ there is an open covering $\mathcal{A}_k \in \mathbb{A}$ whose (open) subsets all have diameter less than $1/k$, then a Lebesgue number argument implies that $\mathbb{A}$ is cofinal in $\mathsf{FinCov}(X)$.

Given a cofinal subobject B and an inverse system F on A, then there is an associated inverse system $F|B$. The following crucial observation suggests the importance an usefulness of such restricted inverse systems.

PROPOSITION 4. *Suppose that we are given the setting above, and let B be a cofinal subobject. Then there is a canonical isomorphism from $\lim_{\leftarrow} F$ to $\lim_{\leftarrow} F|B$.*

Proof. By definition, the inverse limit L_A over all of A is a submodule of $P_A = \prod_{a \in A} F(a)$ and the inverse L_B limit over B is a submodule of $P_B = \prod_{b \in B} F(b)$. Let $\varphi_0 : P_A \to P_B$ be given by the projections onto the factors $F(b)$; since the operations in the product are defined coordinatewise, it follows immediately that φ_0 is a module homomorphism.

By construction it follows that φ_0 maps L_A to L_B. If $\varphi : L_A \to L_B$ be the homomorphism defined by φ_0, the objective is to prove that φ is an isomorphism. It is straightforward to verify that φ is onto. Suppose now that we are given $x = (x_a)$ and $y = (y_a)$ such that $\varphi(x) = \varphi(y)$. Then $x_b = y_b$ for all $b \in B$, and we need to show that this implies $x_a = y_a$ for all a. Let $\alpha \in A$ be arbitrary, and choose $\beta \in B$ such that $\beta \prec \alpha$. Then we have $x_\alpha = f_{\beta,\alpha}(x_\beta)$ and $y_\alpha = f_{\beta,\alpha}(y_\beta)$. Since we are assuming that $y_\beta = x_\beta$, it follows that $y_\alpha = x_\alpha$.■

Definition and properties of Čech homology

Suppose that X is a compact Hausdorff space, let $A \subset X$ be a closed subspace, and let $\mathsf{FinCov}(X, A)$ denote the codirected set of all pairs $(\mathcal{U}, \mathcal{U}|A)$, where $\mathcal{U}$ is a finite open covering of X and $\mathcal{U}|A$ denotes its restriction to A with all empty intersections deleted; the binary relation

$$\beta \quad = \quad (\mathcal{V}, \mathcal{V}|A) \quad \prec \quad (\mathcal{U}, \mathcal{U}|A) \quad = \quad \alpha$$

is taken to mean that $(\mathcal{V}, \mathcal{V}|A)$ is an indexed refinement of $(\mathcal{U},\mathcal{U}|A)$. Since we are working with indexed refinements, it follows that the map of indexing sets will define a simplicial mapping of nerve pairs

$$j_{\beta,\alpha} : (N_\beta, N'_\beta) \quad = \quad \Big(\mathfrak{N}(\mathcal{V}), \mathfrak{N}(\mathcal{V}|A)\Big) \quad \longrightarrow \quad \Big(\mathfrak{N}(\mathcal{U}), \mathfrak{N}(\mathcal{U}|A)\Big) \quad = \quad (N_\alpha, N'_\alpha)$$

and therefore we obtain an inverse system of simplicial complex pairs and simplicial mappings. If we take the simplicial or singular chain complexes associated to such a system we obtain inverse systems of chain complexes, and if we pass to homology we obtain inverse systems of homology groups; at the chain complex level the inverse systems are different, but their homology groups are the same.

Definition. If X is a compact Hausdorff space and $A \subset X$ is a closed subspace, then the **Čech homology groups** $\check{H}_q(X, A)$ are the inverse limits of the inverse systems $H_q(N_\alpha, N'_\alpha)$, where α runs through all pairs $(\mathcal{U},\mathcal{U}|A)$.

Presumably we have introduced these groups because they have implications for dimension theory, and one can also ask if these groups can be computed for finite simplicial complexes. The next two results confirm these expectations.

THEOREM 5. *If X is a compact Hausdorff space whose Lebesgue covering dimension is $\leq n$ and A is a closed subset of X, then $\check{H}_q(X, A) = 0$ for all $q > n$.*

Proof. The condition on the Lebesgue covering dimension implies that every finite open covering $\mathcal{U}$ of X has a (finite) refinement such that each subcollection of $n + 2$ open subsets from $\mathcal{U}$ has an empty intersection. This condition means that the nerve of $\mathcal{U}$ has no simplices with $n + 2$ vertices and hence no simplices of dimension $\geq n + 1$; in other words, the (geometric) dimension of the nerve is at most n. By Proposition 4 and the assumption on the Lebesgue covering dimension, we know that the Čech homology of (X, A) can be computed using open coverings for which each subcollection of $n+2$ open subsets from $\mathcal{U}$ has an empty intersection, and hence the Čech homology is an inverse limit of homology groups of simplicial complexes with dimension $\leq n$. Since the q-dimensional homology of such complexes vanishes if $q > n$, it follows that the same is true for the inverse limit groups when $q > n$, and therefore we must have $\check{H}_q(X, A) = 0$ for all $q > n$.■

The next main result states that the Čech homology for a simplicial complex pair is the same as the homology we have already defined. a more general result:

THEOREM 6. *If X is a compact Hausdorff space and $A \subset X$ is a closed subspace, then there is a canonical mapping φ_∞ from $H_*(X, A)$ to $\check{H}_*(X, A)$ (the* singular-Čech comparison map*), where the groups on the left are singular homology groups. If X is a polyhedron with some simplicial $\mathbf{K}$ such that A is a subcomplex with respect to this decomposition, then the singular-Čech comparison map is an isomorphism.*

Before proving this result, we shall use the conclusion to derive the main implications for dimension theory.

THEOREM 7. *(i) For all $n \geq 0$, the Lebesgue covering dimension of the disk D^n is equal to n.*

(ii) If $(P, \mathbf{K})$ is a simplicial complex whose geometric definition is equal to n, then the Lebesgue covering dimension of P is also equal to n.

(iii) If $A \subset \mathbb{R}^n$ is a compact subset with a nonempty interior, then the Lebesgue covering dimension of A is equal to n.

(*iv*) *If* $\mathbf{Q} = [0,1]^\infty$ *is the Cartesian product of countably infinitely many copies of the unit interval (the so-called* Hilbert cube*), then the Lebesgue covering dimension of* $\mathbf{Q}$ *is equal to* ∞.

Proof. We shall take these in order.

Proof of (*i*). By the discussion at the beginning of this section (or the corresponding discussion in Munkres), we know that the Lebesgue covering dimension of D^n is at most n, so we need to show that it cannot be $\leq (n-1)$. We shall exclude this by deriving a contradiction from it. If the Lebesgue covering dimension was strictly less than n, then it would follow that $\check{H}_n(D^n, A)$ would vanish for all closed subsets $A \subset D^n$. By Theorem 6 we know that $\check{H}_n(D^n, S^{n-1}) \cong H_n(D^n, S^{n-1})$, and since the latter is isomorphic to $\mathbb{Z}$ it follows from Theorem 5 that the Lebesgue covering dimension cannot be $\leq n-1$. Therefore this dimension must be equal to n.■

Proof of (*ii*). This follows immediately from (*i*) and Theorem 50.2 of Munkres (see page 307 for details).■

Proof of (*iii*). By the discussion at the beginning of this section we know that the Lebesgue covering dimension of A is $\leq n$. Since A has a nonempty interior, it follows that A contains a closed subset which is homeomorphic to D^n. This means that the Lebesgue covering dimension of A must be at least as large as the Lebesgue covering dimension of D^n, which is n. Combining these observations, we conclude that the Lebesgue covering dimension of A is equal to n.■

Proof of (*iv*). Let $H\langle n\rangle \subset \mathbf{Q}$ be the subset of all points whose coordinates satisfy $x_k = 0$ for $k \geq n+1$. Then it follows that $H\langle n\rangle$ is a closed subset of $\mathbf{Q}$ which is homeomorphic to D^n, and therefore we have $n = \dim H\langle n\rangle \leq \dim \mathbf{Q}$ for all n.■

Remark. The preceding result implies that the Lebesgue covering dimension does not behave well with respect to quotients, even if the space and its quotient are polyhedra. In particular, if $f : X \to Y$ is a continuous and onto mapping of compact Hausdorff spaces, then in general we cannot say anything about the relation between the Lebesgue covering dimensions of X and Y even if we know that both numbers are finite. The simplest counterexamples are given by the continuous surjection from $[0,1]$ to $[0,1]^2$ given by the Peano curve (described in Section 44 of Munkres) and the usual first coordinate projection from $[0,1]^2$ to $[0,1]$; in the first case the dimension increases when one passes to the quotient, and in the second case the dimension decreases (which is what one reasonably expects). Of course, if we take f as above to be an identity map, then the dimension does not change.

We shall discuss the behavior of dimensions under taking products after proving Theorem 6.

Proof of Theorem 6. We begin by proving the general statement. If $\mathcal{U}$ is an open covering of X and A is a closed subset of X, then we have seen that a partition of unity subordinate to $\mathcal{U}$ defines a canonical map from X into the nerve $\mathfrak{N}(\mathcal{U})$, and by construction this map sends A into $\mathfrak{N}(\mathcal{U}|A)$. We have also seen that the homotopy class of this map is well defined (at least when $A = \emptyset$, but the same argument implies that the canonical maps of pairs associated to different partitions of unity will be homotopic as maps of pairs). Therefore we have homomorphisms

$$(k_\alpha)_* : H_*(X, A) \longrightarrow H_*\big(\mathfrak{N}(\mathcal{U}), \mathfrak{N}(\mathcal{U}|A)\big)$$

and we need to show that these yield a map into the inverse limit of the groups on the right hand side, which is true if and only if

$$(k_\alpha)_* \quad = \quad (j_{\beta\alpha})_* \circ (k_\beta)_*$$

for all α and β such that $\beta \prec \alpha$. But if the latter holds, then it follows that the composite $j_{\beta\alpha} \circ (k_\beta)$ defines a canonical map into the nerve pair (N_α, N'_α), and therefore this composite is homotopic to

k_α; therefore the associated maps in homology are equal, and this implies that we have the desired homomorphism φ_∞ into the inverse limit $\check{H}_*(X, A)$.

We must now show that the singular-Čech comparison map φ_∞ is an isomorphism if X is a polyhedron with simplicial decomposition $\mathbf{K}$ and A corresponds to a subcomplex of $(X, \mathbf{K})$. Let $r > 0$, and let $\mathcal{W}_r$ be the open covering by open stars of vertices in the r^{th} barycentric subdivision $B^r(\mathbf{K})$. Then by construction we have $\mathcal{W}_{r+1} \prec \mathcal{W}_r$ for all r, and a Lebesgue number argument shows that the set of all open coverings $\mathcal{W}_r$ determines a cofinal subset of $\mathsf{FinCov}\,(X)$. If (N_r, N'_r) denotes the nerve pair associated to $\mathcal{W}_r$, then it follows that $\check{H}_*(X, A)$ is isomorphic to the inverse limit of the groups $H_*(N_r, N'_r)$.

If we can show that the canonical maps k_r into N_r all define isomorphisms from $H_*(X, A)$ to $H_*(N_r, N'_r)$, then the map into the inverse limit will be an isomorphism for the following reasons:

(1) If $\varphi_\infty(u) = 0$, then $(k_r)_*(u) = 0$ for all r, and since each of these maps is an isomorphism it follows that $u = 0$.

(2) If v lies in the inverse limit, then v has the form $(v_1, v_2, \cdots)$ where $v_r = (j_{r,r+1})_*(v_{r+1})$ for all r. Since k_r defines an isomorphism, it follows that $v_r = (k_r)_*(u_r)$ for some unique $u_r \in \check{H}(X, A)$, and if we can show that $u_r = u_{r+1}$ for all r then it will follow that $v = \varphi_\infty(u)$. But the previous equations imply that

$$(k_r)_*(u_{r+1}) \quad = \quad (j_{r,r+1})_* \circ (k_{r+1})_*(u_{r+1}) \quad = \quad (j_{r,r+1})_*(v_{r+1}) \;\; v_r \quad = \quad (k_r)_*(u_r)$$

and since $(k_r)_*$ is injective it follows that $u_{r+1} = u_r$.

To conclude the proof, we note that the relative version of Proposition 2 implies that the map of pairs determined by each k_r is homotopic to a homeomorphism of pairs.■

As noted before, this concludes the proof that the Lebesgue covering dimension of D^n is equal to n. It is also possible to prove the following result:

THEOREM 8. *For every $n \geq 0$ the Lebesgue covering dimension of $\mathbb{R}^n$ is equal to n.*

Sketch of proof. The exercises at the end of Section 50 in Munkres (see pages 315–316) provide machinery for extending results on covering dimensions to "reasonable" noncompact spaces. In particular, Exercise 8 shows that the Lebesgue covering dimension of $\mathbb{R}^n$ is at most n. Since the dimension of the closed subspace D^n is equal to n, it follows that the Lebesgue covering dimension of $\mathbb{R}^n$ is at least n, and therefore it must be exactly n.■

One cap proceed similarly to extend the conclusions for Exercises 9 and 10 on page 316 of Munkres. Specifically, *every (second countable) topological n-manifold has Lebesgue covering dimension equal to n, and if $A \subset \mathbb{R}^n$ is a close subset with nonempty interior, then the Lebesgue covering dimension of A is also equal to n.*■

Note. For topological n-manifolds, second countability is equivalent to the σ-compactness condition which appears on page 316 of Munkres (proof?).

Dimensions of products

The standard homeomorphism $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{m+n}$ strongly suggests the following question:

QUESTION. *If we know that the Lebesgue covering dimensions of the nonempty compact Hausdorff spaces X and Y are m and n respectively, does it follow that the Lebesgue covering dimension of the product $X \times Y$ is equal to $m + n$?*

In the next subheading we shall prove the following result:

PROPOSITION 9. *If X and Y are compact Hausdorff spaces whose Lebesgue covering dimensions are m and n respectively, then the Lebesgue covering dimension of the product $X \times Y$ is less than or equal to $m + n$.*

We shall derive this result as an immediate consequence of Proposition 18 below.

If we assume that our spaces are somewhat reasonable, then we can prove a stronger and more satisfying result:

PROPOSITION 10. *In the setting of Proposition 9, suppose that $X = \cup_i A_i$ and $Y = \cup_j B_j$ where the sets A_i and B_j are all homeomorphic to k-disks for suitable values of k. Then the Lebesgue covering dimension of $X \times Y$ is equal to $m + n$.*

Proof of Proposition 10. By Theorem 50.2 of Munkres and finite induction, it follows that the dimension of $X \times Y$ is equal to the maximum of the dimensions of the closed subsets $A_i \times B_j$. On the other hand, the same result implies that there are some indices p and q such that A_p is homeomorphic to D^m and B_q is homeomorphic to D^n (otherwise the dimensions of X and Y would be strictly less than m and n). Since $D^m \times D^n$ is homeomorphic to D^{m+n} it follows that $X \times Y$ has a closed subset with Lebesgue covering dimension equal to $m + n$. On the other hand, we also know that the dimension of each disk A_i is at most m and the dimension of each disk B_j is at most n, so the dimension of $X \times Y$ is at most $m + n$. If we combine these, we find that the dimension of $X \times Y$ is equal to $m + n$.■

Counterexamples to the general question

Although Propositions 9 and 10 may suggest that the formula $\dim(X \times Y) = \dim X + \dim Y$ holds more generally, it is possible to construct examples where the left hand side is less than the right. The first examples of this sort were discovered by L. S. Pontryagin; here is a reference to the original paper:

> **L. S. Pontryagin.** *Sur une hypothèse fondementale de la théorie de la dimension.* Comptes Rendus Acad. Sci. (Paris) **190** (1930), 1105–1107.

In Pontryagin's example one has $X = Y$ and $\dim X = 2$ but $\dim(X \times X) = 3$. By the following result, these are the lowest dimensions in which one can have $\dim(X \times Y) < \dim X + \dim Y$.

DIMENSION ESTIMATES FOR PRODUCTS. *Let X and Y be nonempty compact metric spaces. Then the following hold:*

(*a*) *If* $\dim Y = 0$, *then* $\dim X \times Y = \dim X$.

(*b*) *If* $\dim Y = 1$, *then* $\dim X \times Y = \dim X + 1$.

(*c*) *If* $\dim Y \geq 2$, *then* $\dim X \times Y \geq \dim X + 1$.

Proofs of these results are beyond the scope of this course, so we shall limit ourselves to mentioning some key points which arise in the proofs.

The proof of the first statement is actually fairly direct, and it only requires a small amount of additional machinery. Proofs of the second and third statements using an alternate approach to defining topological dimensions (the *weak inductive* or *Menger-Urysohn dimension*) are due to Hurewicz (we should note that the Menger-Urysohn definition is the one which appears in Hurewicz and Wallman). Here is a reference to the original paper.

W. Hurewicz. *Sur la dimension des produits cartésiens.* Annals of Mathematics **36** (1935), 194–197.

There is a brief indication of another way to retrieve (b) at the top of page 241 in the book by Nagami (however, this requires a substantial amount of input from algebraic topology). One proof of (c) can be obtained by combining (b) with the following existence theorem: *If Y is a compact metric space such that $n = \dim Y$ is finite and $0 < k < n$, then there is a closed subset $B \subset Y$ such that $\dim B = k$.* — This result and the equivalence of the Lebesgue and Menger-Urysohn dimensions for compact metric spaces are discussed in an appendix to this section.

Spaces for which $\dim(X \times Y) < \dim X + \dim Y$ are generally far removed from the sorts of objects studied in most of topology, but it is important to recognize their existence. On the other hand, even though there is no general product formula for the dimensions of compact metric spaces, the validity of the formula for many well-behaved examples (see Proposition 9) leads one naturally to look for necessary and sufficient conditions under which one has $\dim(X \times Y) = \dim X + \dim Y$. Here is one reference which answers the question:

Y. Kodama. *A necessary and sufficient condition under which* $\dim(X \times Y) = \dim X + \dim Y$. Proc. Japan. Acad. **36** (1960), 400–404.

As in several previously cited cases, the proofs of the main results in this paper rely heavily on input from algebraic topology.

Further results

We shall consider two issues related to the discussion of dimension theory:

1. Giving an example of a compact subset of $\mathbb{R}^2$ for which the singular and Čech homology groups are not isomorphic.

2. Showing that a compact subset of $\mathbb{R}^n$ has Lebesgue covering dimension n if and only if it has a nonempty interior (one can then use the previously cited exercises in Munkres to show that the same conclusion holds for arbitrary closed subsets). The machinery developed for this question will also yield a proof of Proposition 9 on the Lebesgue covering dimensions of cartesian products.

The example for the first problem will be the *Polish circle*, and our discussion will be based upon the following online reference:

`http://math.ucr.edu/~res/math205B/polishcircle.pdf`

The key to studying the Čech homology of arbitrary compact subsets in $\mathbb{R}^n$ is a fundamental *continuity property* which does not hold in singular homology.

Continuity in Čech homology

The results in Chapter IX of Eilenberg and Steenrod show that Čech homology is functorial with respect to continuous maps of compact Hausdorff spaces. Given this, we can the basic result very simply.

THEOREM 11. (Continuity Property) *Suppose that X is a subspace of some Hausdorff topological space E, and suppose further that there are compact subsets $X_\alpha \subset E$ such that $X = \cap_\alpha X_\alpha$ for all α and the family X_α is closed under taking finite intersections. Then we have*

$$\check{H}_*(X) \quad \cong \quad \lim_{\leftarrow} \ \check{H}_*(X_\alpha) \ .$$

If $E = \mathbb{R}^n$ for some n, then it is always possible to find such a family of compact subsets X_n such that $X_{n+1} \subset X_n$ for all n and X_n is a finite union of hypercubes of the form

$$\prod_{i=1}^{n} \left(x_i,\, x_i + \frac{1}{2^n} \right)$$

where each x_i is a rational number expressible in the form $p_i/2^n$ for some integer p_i. For example, one can take X_n to be the union of all such cubes which have a nonempty intersection with X.

Reference for the proof of Theorem 11. A proof is given on page 261 of Eilenberg and Steenrod (specifically, see theorem X.3.1).■

Remark. One can also make the singular-Čech comparison map into a natural transformation of covariant functors, but we shall not do this here because it is not needed for our purposes except for a remark following the proof of Theorem 15 (as before, details may be found in Chapters IX and X of Eilenberg and Steenrod).

Singular and Čech homology of the Polish circle

As in the previously cited document

`http://math.ucr.edu/~res/math205B/polishcircle.pdf`

the Polish circle P is defined to be the union of the following curves:

(1) The graph of $y = \sin(1/x)$ over the interval $0 \le x \le 1$.

(2) The vertical line segment $\{1\} \times [-2, 1]$.

(3) The horizontal line segment $[0, 1] \times \{-2\}$.

(4) The vertical line segment $\{0\} \times [-2, 1]$.

One important fact about the Polish circle is that it is arcwise connected but not locally arcwise connected. The proof of this is analogous to the discussion on page 66 of the online notes

`http://math.ucr.edu/~res/math205A/gentopnotes2008.pdf`

which shows that the space B, which is given by closure (in $\mathbb{R}^2$) of the graph of $\sin(1/x)$ for $x > 0$, is connected but not arcwise connected. For the sake of completeness, we shall indicate how one modifies the argument to show the properties of P stated above. First of all, since P is the union of four arcwise connected subspaces $A \cup B \cup C \cup D$ such that $A \cap B$, $B \cap C$ and $C \cup D$ are all nonempty, the arcwise connectedness of P follows immediately. To prove that P is not arcwise connected, we need the following result, whose proof is similar to the previously cited argument which shows that B is not arcwise connected:

LEMMA 12. *Let Y be a compact, arcwise connected, locally arcwise connected topological space, let $f : Y \to P$ be continuous, and suppose that $a_0 \in Y$ is such that the first coordinate of $f(a_0)$ is zero and $f(a_0) \neq (0, -2)$. Then there is an arcwise connected open neighborhood V of a_0 in Y such that $f[V]$ is contained in the intersection of Y with the $y-$axis.*■

This observation has far-reaching consequences for the fundamental group and singular homology of P, all of which come from the following:

PROPOSITION 13. *Let Y and f be as in the preceding lemma. Then there is some $\varepsilon > 0$ such that $f[Y]$ is disjoint from the open rectangular region $(0, \varepsilon) \times (-2, 2)$.*

In terms of the presentation of P given above, this means that $f[Y]$ is contained in the union of $B \cup C \cup D$ with the graph of $\sin(1/x)$ over the interval $[\varepsilon, 1]$. This subspace M_ε is homeomorphic to a closed interval and as such is contractible. Therefore Proposition 13 has the following application to the algebraic-topological invariants of the Polish circle:

THEOREM 14. *If P is the Polish circle, then $\pi_1(P, p)$ is trivial for all $p \in P$, and the inclusion of $\{p\}$ in P induces an isomorphism of singular homology groups.*

Proof of Theorem 14, assuming Proposition 13. We begin with the result on the fundamental group. Suppose that γ is a closed curve in P based at p. By Proposition 13 we know that the image of γ lies in M_ε for some $\varepsilon > 0$, so that the class of γ in $\pi_1(P, p)$ lies in the image of $\pi_1(M_\varepsilon, p)$. Since M_ε is contractible, it follows that the image of $\pi_1(M_\varepsilon, p)$ in $\pi_1(P, p)$ is trivial, and therefore the latter must also be trivial.

The proof for singular homology is similar. If $z \in S_q(P)$ is a cycle, then there is some M_ε such that $p \in M_\varepsilon$ and z lies in the image of $S_q(M_\varepsilon)$. Of course, this means that the class u represented by z lies in the image of the homomorphism $H_q(M_\varepsilon) \to H_q(P)$, and since M_ε is contractible it follows that this image is trivial if $q > 0$. On the other hand, if $q = 0$, then the arcwise connectedness of all the spaces implies that the various inclusion maps all induce isomorphisms in 0-dimensional singular homology.■

Proof of Proposition 13. Let E denote the inverse image of the intersection of P with $\{0\} \times [-\frac{3}{2}, 1]$. Then for each $c \in E$ there is an arcwise connected open neighborhood V_c of c in Y such that $f[V_c]$ is contained in the intersection of Y with the $y-$axis. Let W_c be an open neighborhood of c whose closure is contained in V_c. By continuity E is closed in Y and hence E is a compact subset, so there is a finite subcollection of the sets W_c, say $\{W_1, \cdots, W_n\}$, which covers E.

Define $G \subset Y$ to be the closed subset

$$Y \; - \; \cup_{i=1}^{n} \, W_i$$

so that $f[G]$ is compact and disjoint from $P \cap \{0\} \times [-\frac{3}{2}, 1]$. If $A \subset P$ is the piece of the graph of $\sin(1x)$ described above, then it follows that the second coordinates of all points in $f[G] \cap A$ are positive and by compactness must be bounded away from zero; in other words, there is some $\varepsilon > 0$ such that $f[G] \cap A$ is disjoint from $(0, \varepsilon) \times \mathbb{R}$. But this means that

$$f[Y] \quad = \quad f[G] \; \cup \; \left(\cup_{i=1}^{n} \, f[W_i] \right)$$

must be disjoint from $(0, \varepsilon) \times (-2, 2)$.■

In contrast to the preceding, we have the following result:

THEOREM 15. *The Čech homology groups of the Polish circle P are given by $\check{H}_q(P) = \mathbb{Z}$ if $q = 0, 1$ and zero otherwise.*

The results on Čech homology groups in Eilenberg and Steenrod show that these groups are functorial for continuous mappings and that homotopic mappings induce the same algebraic homomorphisms in Čech homology. If we combine this with Theorem 15 and the results on singular homology, we see that *the Polish circle P is a space which is simply connected and has the singular homology of a point, but P is* `not` *a contractible space.* A self-contained proof of the preceding statement is given in `polishcircle.pdf`.

Proof. We shall prove this using the continuity property of Čech homology as stated above, and we shall use the presentation of P as an intersection of the decreasing closed subsets B_n in the previously cited `polishcircle.pdf`. Since $P = \cap_n B_n$ it follows that

$$\check{H}_* \cong \lim_{\leftarrow} \check{H}_*(B_n)$$

and since each B_n is homeomorphic to a finite simplicial complex (describe this explicitly — it is fairly straightforward), we can replace Čech homology with singular homology on the right hand side. It will suffice to prove that each B_n is homotopic to a circle and the inclusion mappings $B_{n+1} \subset B_n$ are all homotopy equivalences. We shall do this using the subspaces C_n from the `polishcircle` document.

By construction, C_n is a subset of B_n, and we claim that C_n is a deformation retract of B_n. Let X_n be the closed rectangular box

$$\left[\frac{2}{(4n+3)\pi}\right] \times [-1, 1]$$

(the piece shaded in blue in the third figure of `polishcircleA.pdf`), and let Q_n denote the bottom edge of X_n defined by the equation $y = -1$. It follows immediately that Q_n is a strong deformation retract of X_n; since the closure of $B_n - X_n$ intersects X_n in the two endpoints of Y_n, we can extend the retract $X_n \to Y_n$ and homotopy $X_n \times [0,1] \to X_n$ by taking the identity on $\overline{B_n - X_n}$ to extend the retraction and the trivial homotopy from the identity to itself on $\overline{B_n - X_n}$. This completes the proof that C_n is a strong deformation retract of B_n.

By construction the space C_n is homeomorphic to the standard unit circle, and furthermore it is straightforward to check that the composite

$$C_{n+1} \subset B_{n+1} \subset B_n \longrightarrow C_n$$

(where the last map is the previously described homotopy inverse) must be a homeomorphism which is the identity off the points which lie in the vertical strip

$$\left(\frac{2}{4n+7}, \frac{2}{4n+3}\right) \times \mathbb{R}$$

and on this strip it is the flattening map which sends a point $(x, y) \in C_{n+1}$ to $(x, -1) \in C_n$. Therefore the map in homology from $H_q(C_{n+1})$ to $H_q(C_n)$ is an isomorphism of infinite cyclic groups in dimensions 0 and 1 and of trivial groups otherwise, and it follows that the map from $H_q(B_{n+1})$ to $H_q(B_n)$ is also an isomorphism of of infinite cyclic groups in dimensions 0 and 1 and of trivial groups otherwise. As in the proof of the second half of Theorem 6, it follows that $\check{H}_q(X)$ must be infinite cyclic if $q = 0$ or 1 and trivial otherwise.∎

In fact, as noted before the proof of Theorem 15 one can show that a standard map from P to S^1 induces isomorphisms in Čech homology. This requires the naturality property of the comparison map from singular to Čech homology.

Dimensions of nowhere dense subsets

We have seen that if A is a compact subset of $\mathbb{R}^n$ with a nonempty interior, then the Lebesgue covering dimension of A is equal to n; we shall conclude this section with a converse to this result.

In order to prove the converse we shall need some refinements of the ideas which arise in the proof of the embedding theorem stated as Theorem 50.5 in Munkres (see pages 311–313).

Definition. Let $(X, \mathbf{d})$ be a metric space, let $f : X \to Y$ be a continuous map of topological spaces, and let $\varepsilon > 0$. We shall say that f is an $\varepsilon-$*map* if for all $u, v \in X$ the equation $f(u) = f(v)$ implies that $\mathbf{d}(u, v) \le \varepsilon$; an equivalent formulation is that for all $y \in Y$ the diameter of the level set $f^{-1}[\{y\}]$ is less than or equal to ε.

Clearly a continuous map f is 1–1 if and only if it is an $\varepsilon-$map for all $\varepsilon > 0$ (equivalently, it suffices to have this condition for all numbers of the form $1/k$ where k is a positive integer or all numbers of the form 2^{-k} where k is a positive integer).

We shall need the following result, which is entirely point set-theoretic.

LEMMA 17. *Let $(X, \mathbf{d}_X)$ and $(Y, \mathbf{d}_Y)$ be compact metric spaces, let $\varepsilon > \varepsilon' > 0$, and let $f : X \to Y$ be a continuous $\varepsilon'-$map. Then there is a $\delta > 0$ such that if $A \subset Y$ has diameter less than or equal to δ, then $f^{-1}[A]$ has diameter less than ε.*

Proof. Let $\eta = \frac{1}{2}(\varepsilon + \varepsilon')$ and let $K_\eta \subset X \times X$ be the set of all (x_1, x_2) such that $\mathbf{d}_X(x_1, x_2) \ge \eta$. Then K_η is a closed (hence compact) subset of $X \times X$ and $f \times f[K_\eta]$ is a compact subset of $Y \times Y$ which is disjoint from the diagonal Δ_Y because f is an $\varepsilon'-$map. It follows that the restriction of the distance function $\mathbf{d}_Y$ to $f \times f[K_\eta]$ is bounded away from zero by a positive constant h; in other words, if $U_h \subset Y \times Y$ is the set of all $(y_1, y_2) \in Y \times Y$ such that $\mathbf{d}_Y(y_1, y_2) \le h/2$, then $(y_1, y_2) \notin f \times f[K_\eta]$.

Suppose now that the diameter of A is less than $\delta = h/2$; then we have $A \times A \subset U_h$, and it follows that if $(p, q) \in f^{-1}[A]$, then $\mathbf{d}_Y\big(f(p), f(q)\big) < \delta$, and this means that (p, q) cannot lie in K_η because the image of the latter under $f \times f$ is disjoint from U_h, which contains $A \times A$. In other words, if the diameter of A is less than δ, then the diameter of $f^{-1}[A]$ must be less than or equal to η, which is less than ε.■

The next result gives a method for approximating n-dimensional compact metric spaces by n-dimensional simplicial complexes.

PROPOSITION 18. *Let X be a compact metric space, and let n be a nonnegative integer. Then the Lebesgue covering dimension of X is $\le n$ if and only if for every $\varepsilon > 0$ there is an $\varepsilon-$map from X into some n-dimensional polyhedron P.*

Proof. Suppose first that the Lebesgue covering dimension of X is $\le n$. Take the open covering of X by open disks of radius $\varepsilon/2$ about the points of X, and extract a finite subcovering

$$\mathcal{U} \;=\; \left\{ N_{\varepsilon/2}(x_1),\ \cdots\ ,\ N_{\varepsilon/2}(x_m) \right\} .$$

Let $\{ \varphi_j \}$ be a partition of unity subordinate to this finite covering, and consider the canonical map k from X to $\mathfrak{N}(\mathcal{U})$. If $k(u) = k(v)$, then $\varphi_j(u) = \varphi_j(v)$ for all j; at least one of these values must be positive, and therefore we can find some j such that $u, v \in N_{\varepsilon/2}(x_j)$. Since the latter implies $\mathbf{d}(u, v) \le \text{diameter} N_{\varepsilon/2}(x_j) \le \varepsilon$, it follows that k is an $\varepsilon-$map.

As usual, with respect to this metric there is a Lebesgue number $\eta > 0$ for this open covering. Let $0 < \varepsilon' < \varepsilon < \eta$, and let $f : X \to P$ be an $\varepsilon'-$map from X to some polyhedron P of dimension $\le n$. By the preceding lemma there is some $\delta > 0$ such that if $A \subset Y$ has diameter less than δ then $f^{-1}[A]$ has diameter less than ε.

Take a sufficiently large barycentric subdivision of P such that all simplices have diameter at most $\delta/2$, and let $\mathcal{V}$ be the open covering given by the inverse images (under f) of open stars of the vertices in P. Then the intersection of any $n+2$ open subsets in $\mathcal{V}$ is empty; if we can show that $\mathcal{V}$ is a refinement of $\mathcal{U}$, then we are done. But the open stars of vertices in P all have diameter at most δ, and thus by Lemma 17 their inverse images have diameters which are at most ε. Since ε is less than a Lebesgue number for $\mathcal{U}$, it follows that each of the open subsets in $\mathcal{V}$ must be contained in some open set from $\mathcal{U}$, and thus $\mathcal{V}$ is an open refinement of $\mathcal{U}$ such that every subcollection $n+2$ subsets in $\mathcal{V}$ has an empty intersection.■

Before proceeding, we shall show that Proposition 18 yields the previously stated result about the dimensions of Cartesian products (namely, $\dim(X \times Y) \leq \dim X + \dim Y$). In this argument we assume that $\dim X$ and $\dim Y$ are both finite; it is straightforward to verify that if X and Y are $\mathbf{T_1}$ spaces and either $\dim X = \infty$ or $\dim Y = \infty$, then $\dim(X \times Y) = \infty$ (look at the contrapositive statement).

Proof of Proposition 9. Suppose that $\dim X \leq m$ and $\dim Y \leq n$, and let $\varepsilon > 0$. By Proposition 18, it will suffice to construct an ε-map from $X \times Y$ to some polyhedron T of dimension at most $m+n$. For the sake of definiteness, in this argument the metrics on products are given by the $\mathbf{d}_2$ metrics associated to metrics on the factors (using the notation of the 205A notes).

The construction is fairly straightforward. By the dimension hypotheses and Proposition 18 we know there are $(\varepsilon/\sqrt{2})$-maps $f : X \to P$ and $g : Y \to Q$, where P and Q are polyhedra of dimension at most m and n respectively. It follows that the product map $f \times g : X \times Y \to P \times Q$ is an ε-map into a polyhedron whose dimension is at most $m+n$.■

Using Proposition 18, we can prove the result on the dimensions of nowhere dense subsets mentioned above.

THEOREM 19. *Suppose that $A \subset \mathbb{R}^n$ is compact and nowhere dense. Then the Lebesgue covering dimension of A is at most $n-1$.*

The estimate in the theorem is the best possible estimate because we know that the Lebesgue covering dimension of the nowhere dense subset S^{n-1} is equal to $n-1$.

Proof. We shall prove that A satisfies the criterion in Proposition 18. One step in the proof involves the following result:

CLAIM. *If $\mathbf{v}$ is an interior point of the disk D^n where $n > 0$, then S^{n-1} is a retract of $D^n - \{\mathbf{v}\}$.*

The quickest way to prove this is to take the map $\rho : D^n \times D^n - \text{diagonal} \to S^{n-1}$ constructed in the file `brouwer.pdf` and restrict it to $(D^n - \{\mathbf{v}\}) \times \{\mathbf{v}\}$.

The first steps in the proof are to let $\varepsilon > 0$ and to take a large hypercube Q containing A. We know that Q has a simplicial decomposition, and if we take repeated barycentric subdivisions we can construct a decomposition whose simplices all have diameter less than $\varepsilon/2$. Let σ be an n-simplex in this decomposition. Since σ has a nonempty interior (in the sense of point set topology) and A is nowhere dense in $\mathbb{R}^n$, it follows that there is some interior point $w(\sigma)$ in σ such that $w(\sigma) \notin A$. By the claim above, we know that the boundary $\partial\sigma$ is a retract of $\sigma - w(\sigma)$, and we can piece the associated retractions together to obtain a retraction

$$r : Q \; - \; \left(\bigcup_{\dim \sigma = n} \{w(\sigma)\} \right) \; \longrightarrow \; Q^{[n-1]}$$

where $Q^{[n-1]}$ (the *n-skeleton*) is the union of all simplices in Q with dimension strictly less than n. By construction the set A is contained in the domain of r, and therefore we also obtain a retraction $r : A \to Q^{[n-1]}$. The inverse image of a point z in the codomain is contained in all simplices which contain z, and since these simplices all have diameter less than $\varepsilon/2$, it follows that each set $r^{-1}[\{z\}]$ has diameter less than ε. Therefore we have shown that $r|A$ is an $\varepsilon-$map into the $(n-1)$-dimensional polyhedron $Q^{[n-1]}$. By Proposition 18, it follows that the Lebesgue covering dimension of A is at most $n-1$.∎

Using results from Section 50 of Munkres (including the exercises), it is a straightforward exercise to prove the following generalization of Theorem 19:

COROLLARY 20. *Let M^n be a second countable topological n-manifold, and suppose that $A \subset M$ is a closed nowhere dense subset of M^n. Then the Lebesgue covering dimension of A is strictly less than n.*∎

Appendix : The Flag Property

Default hypothesis. Unless stated otherwise, all spaces discussed in this Appendix are compact metric spaces with finite Lebesgue covering dimensions.

In our discussion of product formulas for the Lebesgue covering dimension, we noted that $\dim X \times Y > \dim X$ if $\dim Y > 0$, and we gave references for the proof when $\dim Y = 1$. We also asserted that the general case followed quickly from this special case because $\dim Y > 0$ implies the existence of a closed subset $A \subset Y$ with $\dim A = 1$. In fact, we have the following:

PROPOSITION A1. (Flag Property) *Suppose that X satisfies the Default Hypothesis and $\dim X = n > 0$. Then there is a chain of closed subsets*

$$\{y\} \quad = \quad A_0 \quad \subset \quad A_1 \quad \subset \quad \cdots \quad \subset A_n \quad = \quad X$$

such that $\dim A_k = k$ for all k.

Note. The name for this result is motivatived by a standard geometrical concept of a *flag* of subspaces in $\mathbb{R}^n$, which is a sequence of vector subspaces

$$\{\mathbf{0}\} \quad = \quad V_0 \quad \subset \quad V_1 \quad \subset \cdots \subset \quad V_n \quad = \quad \mathbb{R}^n$$

such that $\dim V_k = k$ for all k; of course, there is a similar concept if $\mathbb{R}$ is replaced by an arbitrary field.

The proof of the Flag Property is a fairly direct consequence of equality of the Lebesgue covering dimension and the previously cited Menger-Urysohn or weak inductive dimension for compact metric spaces. Here is a summary of what we need in order to prove the Flag Property:

THEOREM A2. *Let X be a compact metric space such that $\dim X \leq n$, and let $x \in X$. Then there is a countable neighborhood base at x of the form*

$$\mathfrak{B} \quad = \quad \{\, W_1 \quad \supset \quad W_2 \quad \cdots \qquad \}$$

such that for each k the set $\mathsf{Bdy}_x(W_k)$ has dimension at most $n-1$. Conversely, if such neighborhood bases exist for each point of X, then $\dim X \leq n$.

As in Munkres, the boundary (or frontier) $\mathsf{Bdy}_X(E)$ of $E \subset X$ (in X) is the intersection of the limit point sets $\mathbf{L}_X(E) \cap \mathbf{L}_X(X - E)$; since we are working with metric spaces, this is a closed subset of X.

Idea of proof for Theorem A2. The statement in the conclusion is essentially the same as the condition for the Menger-Urysohn dimension of X to be at most n (this is given on page 24 of Hurewicz and Wallman). Therefore the conclusion will follow if we know that the Lebesgue covering dimension and the Menger-Urysohn dimension are equal for compact metric spaces. Virtually every book on dimension theory from the past 50 years contains some abstract version of this equality. More directly, one can use Theorem V.8 on page 67 of Hurewicz and Wallman (in which "dimension" means the Menger-Urysohn dimension) to show that the two definitions are the same for compact metric spaces.

One reason that the standard references for dimension theory phrase things in more abstract terms is that the Lebesgue covering dimension and Menger-Urysohn dimension are not necessarily equal for more general topological spaces (usually it is easy to find examples; see also the Wikipedia article on inductive dimension mentioned earlier).

Proof of Proposition A1. (Compare Hurewicz and Wallman, Proposition III.1.D, pp. 24–25.) If $\dim X = 1$ then X is nonempty and the conclusion follows immediately. Proceeding by induction on the dimension, we shall assume the result is true for compact metric spaces of dimension $\leq n-1$. Suppose that X is an n-dimensional compact metric space. Since $\dim X$ is not less than or equal to $n-1$, Theorem A2 implies the existence of some point $z \in X$ such that for all countable neighborhood bases at z of the form

$$\mathfrak{A} \quad = \quad \{ V_1 \quad \supset \quad V_2 \quad \cdots \qquad \}$$

we have $\dim(\mathsf{Bdy}_X(V_k)) > n-2$ for infinitely many k (why?). In particular, this holds for the neighborhood base $\mathfrak{B}$ for z described in the statement of Theorem A2 (we know such a neighborhood base exists because $\dim X = n$). It follows that $\dim(\mathsf{Bdy}_X(W_k)) = n-1$ for all such k. Choose a specific m such that $\dim(\mathsf{Bdy}_X(W_m)) = n-1$. By the induction hypothesis, there is a chain of closed subspaces

$$\{y\} \quad = \quad A_0 \quad \subset \quad A_1 \quad \subset \quad \cdots \quad \subset \quad A_{n-1} \quad = \quad \mathsf{Bdy}_X(W_m)$$

and we may extend this to a chain of subspaces as in the conclusion of the proposition by taking $A_n = X$.■

III.6 : Homology and line integrals

(Lee, *Introduction to Smooth Manifolds*, Chs. 6, 12, 14)

See Section VIII.6 in the course directory file `fundgp-notes.pdf`, the exercises for this section in `fundgpexercises2014.pdf`, and the file `disks-with-holes.pdf`. Among other things, these documents discuss some basic questions involving analytic functions of one complex variable. Far-reaching generalizations of the results in these documents are discussed at the end of Unit V.

IV. Singular Cohomology

Suppose that $\mathbb{F}$ is a field and (X, A) is a pair of topological spaces. One can then define the q-dimensional cohomology $H^q(X, A; \mathbb{F})$ to be the vector space dual $\mathrm{Hom}_{\mathbb{F}}\big(H_q(X, A; \mathbb{F}), \mathbb{F} \big)$, and this construction extends to a contravariant functor on the category of pairs of spaces and continuous maps. Similarly, by taking adjoint maps of dual spaces we obtain natural coboundary morphisms $\delta : H^q(A; \mathbb{F}) \to H^{q+1}(X, A; \mathbb{F})$ and long exact cohomology sequences for pairs.

One natural question is why one would bother to do this, especially since it follows that $H_q(X; \mathbb{F}) \cong H^q(X; \mathbb{F})$ if X has the homotopy type of a finite cell complex (because the homology is finite dimensional and is isomorphic to its vector space dual). There are two related answers:

(1) Even when mathematical objects and their duals are equivalent, in many cases it is more convenient to work with the dual object rather than the original one, and vice versa. — For example, vector fields and differential 1-forms on a smooth manifold are dual to each other, but they play markedly different roles in the theory of smooth manifolds. In particular, vector fields are better for working with differential equations, while differential forms provide a more convenient way for manipulating expressions like line integrals.

(2) Frequently the dual objects have some extremely useful extra structure which is not easily studied in the original objects. — To continue with our example of vector fields and differential 1-forms, the latter have better functoriality properties, and the exterior derivative construction on differential forms does not have a functorial counterpart for vector fields unless one adds some further structure like a riemannian metric. On the other hand, there can also be some nice structure on the original objects which is not on their duals; for example, the Lie bracket construction on vector fields has no obvious counterpart on 1-forms unless one adds some further structure.

In fact, it turns out that cohomology groups have a useful additional structure; namely, there are natural bilinear **cup product** mappings

$$\cup \ : \ H^p(X, A; \mathbb{F}) \times H^q(X, A; \mathbb{F}) \ \longrightarrow \ H^{p+q}(X, A; \mathbb{F})$$

which do not have comparably simple counterparts in homology. This illustrates the second point about objects and their duals. Later in these notes we shall illustrate how the first point manifests itself in homology and cohomology.

A useful result

At several points in this unit we shall need the following result on **acyclic** (no homology) chain complexes.

THEOREM 0. *Let C_* be a chain complex such that $C_k = 0$ for $k < M$ for some integer M and each C_k is free abelian on some set of generators G_k. Then $H_*(C) = 0$ in all dimensions if and only if there is a contracting chain homotopy $D_q : C_q \to C_{q+1}$ (for all q) such that $Dd + dD = \mathbf{id}_C$.*

Proof. If D exists then clearly the homology is zero by the usual sort of argument. Conversely, suppose that $H_*(C) = 0$, and let m be the first degree in which C is nonzero. We may construct D_m as follows: If $T \in G_m$, then $dT = 0$ and hence $T = du$ for some $u \in C_{m+1}$. Define $D_m(T) = u$ and extend the map using the freeness property. Now suppose by induction that we have defined D_k for $k \leq N - 1$.

Let T be an element of the free generating set G_N. Then we need to find an element $u_T \in C_{n+1}$ so that $du_T = T - DdT$. Since the complex has no homology, such a class exists if and only if the right hand side is a boundary. But now we have a familiar sort of inductive calculation:

$$d(T - DdT) \ = \ dT - dDdT \ = \ dT - (1 - Dd)dT \ = \ dT - dT - DddT \ = \ -DddT \ = \ D0T \ = \ 0$$

Hence we can define $D(T) = u_T$ and extend by freeness. This completes the inductive step and proves the existence of the contracting chain homotopy.■

REMARK ON THE EXPOSITION. Many of the arguments and constructions in the remaining units of these notes are variations of ideas that were introduced earlier. Partly for this reason, the proofs will often be less detailed than in previous units, with the details left to the reader (also see the earlier quotation from Davis and Kirk on pages 18–19 of these notes and the following quotation from page *vii* of Spanier: The reader is expected to develop facility for the subject as he [or she] progresses, and accordingly, the further he [or she] is in the book, the more he [or she] is called upon to fill in details of proofs).

IV.1 : The basic definitions

(Hatcher, §§ 3.1–3.2)

We begin by defining the singular cohomology of a space with coefficients in an arbitrary $\mathbb{D}$-module, where $\mathbb{D}$ is a commutative ring with unit (a setting broad enough to contain coefficients in fields, the integers, and quotients of the latter). However, we shall quickly specialize to the case of fields in order to minimize the amount of algebraic machinery that is needed.

Definition. Let (X, A) be a pair of topological spaces, and let π be a module over the ring $\mathbb{D}$ as above. The **singular cochain complex** $\big(S^*(X, A; \pi),\ \delta \big)$ of (X, A) **with coefficients in** π is defined with $S^q(X, A) = \mathrm{Hom}\big(S_q(X, A),\ \pi \big)$ and the *coboundary mapping*

$$\delta^{q-1} : S^{q-1}(X, A; \pi) \ \longrightarrow \ S^q(X, A; \pi)$$

given by the adjoint map $\mathrm{Hom}(d_q, \pi)$.

Many basic properties of singular cochain complexes follow immediately from the definitions, including the following:

PROPOSITION 1. *(i) We have* $\delta^q \circ \delta^{q-1} = 0$.

(ii) The singular cochain complex is contravariantly functorial with respect to continuous mappings on pairs of topological spaces.

The first of these follows because $d_q \circ d_{q+1} = 0$ and the functor $\mathrm{Hom}(-, -)$ is additive, while the second is basically just a consequence of the definition and the covariant functoriality of the singular chain complex.■

Before going further, we shall define the q-dimensional **singular cohomology** $H^q(X, A; \pi)$ of (X, A) **with coefficients in** π to be the kernel of δ^q modulo the image of δ^{q-1}. Elements of the kernel are usually called **cocycles**, and elements of the image are usually called **cobooundaries**. As in the case of singular chain complexes, it follows that the map of singular cochains

$$f^{\#} : S^*(Y,B;\pi) \longrightarrow S^*(X,A;\pi)$$

associated to a continuous map $f : (X,A) \to (Y,B)$ will pass to a homomorphism

$$f^* : H^*(Y,B;\pi) \longrightarrow H^*(X,A;\pi)$$

and this makes singular cohomology into a contravariant functor on pairs of spaces and continuous maps.

If (X,A) is a pair of topological spaces, then for each q we know that $S_q(X) \cong S_q(A) \oplus S_q(X,A)$ **as free abelian groups** (but this is **NOT** an isomorphism of chain complexes!), and from this it follows that for each q we have a split short exact sequence of modules

$$0 \longrightarrow S^*(X,A;\pi) \xrightarrow{j^{\#}} S^*(X;\pi) \xrightarrow{i^{\#}} S^*(A;\pi) \longrightarrow 0$$

where $j : X \to (X,A)$ and $i : A \to X$ are the usual inclusions. As in the case of singular chains, this leads to a natural **long exact cohomology sequence**; to simplify the notation we shall omit the coefficient module π in the display below:

$$\cdots \quad H^{k-1}(A) \xrightarrow{\delta} H^k(X,A) \xrightarrow{j^*} H^k(X) \xrightarrow{i^*} H^k(A) \xrightarrow{\delta} H^{k+1}(X,A) \quad \cdots$$

As in the case of homology, this sequence extends indefinitely to the left and right.

Notational convention. The contravariant algebraic maps induced by inclusions are often called **restriction** maps; one motivation for this terminology is that a map like $i^{\#}$ restricts attention from objects defined for X to objects defined only for the subspace A (for example, consider the restriction map from continuous real valued functions on X to those defined on A, which is defined by composing a function $f : X \to \mathbb{R}$ with the inclusion mapping i).

We can now proceed as in the study of singular homology to prove homotopy invariance, excision, and Mayer-Vietoris theorems for singular cohomology; informally speaking, one need only apply the functor $\mathrm{Hom}(-,\pi)$ to everything in sight, including chain homotopies. At some points one needs Theorem 0 to conclude that if C_* is an acyclic, free abelian chain complex, then it has a contracting chain homotopy and the latter implies that $\mathrm{Hom}(C_*,\pi)$ has no nonzero cohomology (verify this!).

Cup products

We shall now assume that our coefficients π are a commutative ring with unit, which we shall call $\mathbb{D}$.

Definition. Let X be a space; then the *augmentation mapping* $\varepsilon_X(\mathbb{D}) = \varepsilon_X \in S^0(X;\mathbb{D})$ is the homomorphism from $S_0(X)$ to $\mathbb{D}$ which sends each singular 0-simplex $T : \Delta_0 \to X$ to the unit element of $\mathbb{D}$.

The following is an immediate consequence of the definitions.

PROPOSITION 2. *If $f : X \to Y$ is continuous, then $f^{\#}(\varepsilon_Y) = \varepsilon_X$. Furthermore, $\delta^0(\varepsilon_X) = 0$.*■

The augmentation plays a key role in the multiplicative structure mentioned earlier. Before proceeding, we need some geometric definitions.

Definition. Let p and q be nonnegative integers, and as usual let Δ_{p+q} denote the standard simplex. Then the *front and back faces* $\mathbf{Front}_p(\Delta_{p+q})$ and $\mathbf{Back}_q(\Delta_{p+q})$ are the p- and q-dimensional faces whose vertices are respectively the first $(p+1)$ and last $(q+1)$ vertices of the original simplex. Note that these intersect in the p^{th} vertex of Δ_{p+q}.

Definition. Given two cochains $f \in S^p(X, A; \mathbb{D})$ and $g \in S^q(X, A; \mathbb{D})$, their **cup product** $f \cup g \in S^{p+q}(X, A; \mathbb{D})$ is given as follows: For each standard free generator of $S_{p+q}(X, A)$ — in other words, each singular simplex of X whose image is not entirely contained in A — we define

$$f \cup g(T) \quad = \quad (f|\mathbf{Front}_p) \cdot (g|\mathbf{Back}_q) \ .$$

We then have the following:

PROPOSITION 3. *The cup product is functorial for continuous maps of pairs. Furthermore, it is bilinear and associative, and if $A = \emptyset$ then ε_X is a two sided multiplicative identity.*$^{(\star)}$■

At this point we do not want to address questions about the possible commutativity properties of the cup product. This is a decidedly nonelementary issue, and in several respects it is a fundamental difficulty which has an enormous impact across most if not all of algebraic topology.

Clearly one would hope the cup product will pass to cohomology, and the following result guarantees this:

PROPOSITION 4. *In the notation of the cup product definition, we have*

$$\delta(f \cup g) \quad = \quad (\delta f) \cup g \ + \ (-1)^p \, f \cup (\delta g) \ .$$

In particular, it follows that $f \cup g$ is a cocycle if both f and g are, and if we are given equivalent representatives f' and g' for the same cohomology classes, then $f \cup g - f' \cup g'$ is a coboundary.

Proof. The identity involving the coboundary of $f \cup g$ is derived in Lemma 3.6 on page 206 of Hatcher$^{(\star)}$. If f and g are both coboundaries, the formula immediately implies that $f \cup g$ is also a coboundary. Suppose now that we also have $f - f' = \delta v$ and $g - g' = \delta w$. It then follows that

$$\delta(v \cup g) \quad = \quad (f - f') \cup g \ , \qquad \delta(f' \cup w) \quad = \quad \pm \, f' \cup (g - g') \ .$$

The first of these implies that $f \cup g$ and $f' \cup g$ determine the same cohomology class, while the second implies that $f' \cup g$ and $f' \cup g'$ also determine the same cohomology class.■

Relative cup products

In many contexts it is useful to have a slight refinement of the cup product described above. Specifically, if A and B are both subspaces of X which satisfy some regularity condition — for example, if both are open in X — then we shall define a relative cup product

$$H^p(X, A; \mathbb{D}) \times H^q(X, B; \mathbb{D}) \ \longrightarrow \ H^{p+q}(X, A \cup B; \mathbb{D})$$

which is a very slight modification of the definition given above.

Suppose first that A and B are open in X, and let $\mathcal{F}$ be the open covering of X given by $\{A, \, B\}$. Let $S_*^{\mathcal{F}}(A \cup B)$ be the subcomplex of $\mathcal{F}$-small singular chains, let $S^*_{\mathcal{F}}(A \cup B)$ be the associated cochain complex, and let $S^*_{\mathcal{F}}(X, A \cup B)$ be the kernel of the restriction map from $S^*(X)$ to $S^*_{\mathcal{F}}(A \cup B)$. Equivalently, $S^*_{\mathcal{F}}(X, A \cup B)$ is the cochain complex associated to the quotient

$$S_*(X)/S_*^{\mathcal{F}}(A \cup B) \quad = \quad S_*(X)/\left(\, S_*(A) + S_*(B) \, \right)$$

and since A and B are open in X, it follows that $S^*_{\mathcal{F}}(X, A \cup B)$ is a quotient of $S^*(X, A)$ such that the projection from $S^*(X, A \cup B) \to S^*_{\mathcal{F}}(X, A \cup B)$ induces isomorphisms in cohomology.

Suppose now that we are given cochains $f \in S^p(X, A)$ and $g \in S^q(X, B)$; by construction $S^p(X, A)$ and $S^q(X, B)$ are cochain subcomplexes of $S^*(X)$, and therefore the cup product construction defines a cochain $f \cup g : S_{p+q}(X) \to \mathbb{D}$. We need to show that this cochain actually lies inside $S^*_{\mathcal{F}}(A \cup B)$, or equivalently that the restriction of $f \cup g$ to $S^{\mathcal{F}}_*(A \cup B) = S_*(A) + S_*(B)$ is trivial. This will follow if we can show that the restrictions of $f \cup g$ to both $S_*(A)$ and $S_*(B)$ are zero, and thus it suffices to show that $f \cup g(T) = 0$ if T is a singular simplex in A or B.

Let T be a singular simplex in A or B; symmetry considerations show it suffices to consider the first case (reverse the roles of the variables to get the other case). Then $f \cup g(T) = f(T_1) \cdot g(T_2)$, where T_i is obtained by restricting T to a front or back face of Δ_{p+q}. If the restriction of f to $S_*(A)$ is zero, then it follows from the previous formula that $f \cup g(T) = 0$. Similarly, if the restriction of g to $S_*(B)$ is zero, then one obtains the same conclusion. Therefore $f \cup g$ actually lies in $S^*_{\mathcal{F}}(A \cup B)$; the previous arguments show that $f \cup g$ is a cocycle if f and g are cocycles and in this case the cohomology class of $f \cup g$ depends only on the cohomology classes of f and g. This gives us a map from $H^p(X, A) \times H^q(X, B)$ to the cohomology of $S^*_{\mathcal{F}}(X, A \cup B)$, and since the surjection from $S^*(X, A \cup B)$ to this group induces cohomology isomorphisms it follows that we obtain a class in $H^{p+q}(X, A \cup B; \mathbb{D})$. This refined cup product has analogs of all the properties one might expect to generalize from the case $A = B$; for example, it is associative.

Simplicial cohomology

As before, let π be an abelian group.

Given a simplicial complex $(P, \mathbf{K})$ and a subcomplex $(Q, \mathbf{L})$, one can define the (unordered) simplicial cochain complex $C^*(\mathbf{K}, \mathbf{L}; \pi)$ to be $\mathrm{Hom}(C_*(\mathbf{K}, \mathbf{L}); \pi)$. These objects are contravariantly functorial with respect to subcomplex inclusions, and as before one obtains long exact cohomology sequences for pairs. Furthermore, if we apply $\mathrm{Hom}(...; \pi)$ to the canonical natural maps $\lambda : C_*(\mathbf{K}, \mathbf{L}) \to S_*(P, Q)$, then we obtain canonical natural cochain complex maps

$$\psi : S^*(P, Q; \pi) \longrightarrow C^*(\mathbf{K}, \mathbf{L}; \pi)$$

and these in turn yield a commutative ladder diagram relating the long exact cohomology sequences for (P, Q) and $(\mathbf{K}, \mathbf{L})$. Previous experience suggests that the associated cohomology maps ψ^* should be isomorphisms, and we shall prove this below.

PROPOSITION 5. *The maps ψ^* define isomorphisms relating the long exact cohomology sequences for (P, Q) and $(\mathbf{K}, \mathbf{L})$.*

Proof. Consider the functorial chain maps λ as above; we known these maps define isomorphisms in homology. By construction λ maps a free generator $\mathbf{v}_0 \ \cdots \ \mathbf{v}_q$ of $C_q(\mathbf{K}, \mathbf{L})$ to an affine singular q-simplex T for (P, Q); therefore, if $\mathbf{V}_*(\mathbf{K}, \mathbf{L})$ is the quotient of $S_*(P, Q)$ by the image of λ, then it follows that the chain group $\mathbf{V}_q(\mathbf{K}, \mathbf{L})$ is free abelian on a subset of free generators for $S_q(P, Q)$, and by the long exact homology sequence for the short exact sequence

$$0 \to C_* \to S_* \to \mathbf{V}_* \to 0$$

it follows that all homology groups of $\mathbf{V}_*(\mathbf{K}, \mathbf{L})$ are zero. We can now use Proposition VI.0 to conclude that $\mathbf{V}_*(\mathbf{K}, \mathbf{L})$ has a contracting chain homotopy D_*, and we can use the associated maps $\mathrm{Hom}(D_*, \pi)$ to conclude that for each π all the cohomology groups of the cochain complex

$\text{Hom}(\mathbf{V}_*, \pi)$ are also zero. If we now apply this observation to the long exact cohomology sequence associated to

$$0 \;\to\; \text{Hom}(\mathbf{V}_*, \pi) \;\to\; \text{Hom}(S_*, \pi) \;\to\; \text{Hom}(C_*, \pi) \;\to\; 0$$

we see that the map $\psi : \text{Hom}(S_*, \pi) \to \text{Hom}(C_*, \pi)$ must also induce isomorphisms in cohomology.■

Given a simplicial complex $(P, \mathbf{K})$ and an ordering of its vertices, one can similarly define an ordered cochain complex $C^*(P, \mathbf{K}^\omega)$ and canonical cochain complex maps

$$\alpha : C^*(P, \mathbf{K}) \;\longrightarrow\; C^*(P, \mathbf{K}^\omega)$$

and an analog of the preceding argument then yields the following result:

COROLLARY 6. *The associated maps in cohomology α^* are isomorphisms.*■

CUP PRODUCTS. If $\mathbb{D}$ is a commuative ring with unit, then one can define cup products on the cochain complexes $C^*(\mathbf{K}, \mathbb{D})$ using the same construction as in the singular case, and it is an elementary exercise to check that (a) this cup product has the previously described properties of the singular cup product, (b) the cochain map ψ preserves cup products at the cochain level (hence also in cohomology)$^{(\star)}$.

Examples of cochains

Formally speaking, cochains are fairly arbitrary objects, so we shall describe some "toy models" which reflect typical and important contexts in which concrete examples arise (also see Exercise VI.2 in `advnotesexercises.pdf`). As usual, let $(P, \mathbf{K})$ be a polyhedron in $\mathbb{R}^n$, and let $f : P \to \mathbb{R}$ be a continuous function. We can then define a (simplicial) **line integral cochain** $\mathbf{L}_f \in C^1(\mathbf{K}; \mathbb{R})$ on free generators $\mathbf{v}_0\mathbf{v}_1$ by the formula

$$\mathbf{L}_f(\mathbf{v}_0\mathbf{v}_1) \;=\; \int_0^1 f\big(t\mathbf{v}_1 + (1-t)\mathbf{v}_0\big)\,|\mathbf{v}_1 - \mathbf{v}_0|\,dt \;\in\; \mathbb{R}\;.$$

By construction, this is just the scalar line integral of f along the directed straight line curve from $\mathbf{v}_0$ to $\mathbf{v}_1$.

Similarly, if $(P, \mathbf{K})$ is a polyhedron in $\mathbb{R}^3$ and $f : P \to \mathbb{R}$ is continuous, then we can define a surface integral cochain $\mathbf{S}_f \in C^2(\mathbf{K}; \mathbb{R})$ by the standard surface integral formula for scalar functions:

$$\mathbf{S}_f(\mathbf{v}_0\mathbf{v}_1\mathbf{v}_2) \;=\; \int_0^1 \int_0^{1-t} f(s\mathbf{v}_1 + t\mathbf{v}_2) \cdot |(\mathbf{v}_1 - \mathbf{v}_0) \times (\mathbf{v}_2 - \mathbf{v}_0)|\,ds\,dt$$

In this formula "$\times$" denotes the usual vector cross product. There are also versions of this construction in higher dimensions which yield cochains of higher dimension, but we shall not try to discuss them here.

Finally, given a field $\mathbb{F}$ we shall construct an explicit example of a cocycle in $C^1(\partial\Delta_2^\omega; \mathbb{F})$ which is not a coboundary.

By construction $C_1(\partial\Delta_2^\omega)$ is free abelian on free generators $\mathbf{e}_i\mathbf{e}_j$, where $0 \le i < j \le 2$. Thus a 1-dimensional cochain f is determined by its three values at $\mathbf{e}_0\mathbf{e}_1$, $\mathbf{e}_0\mathbf{e}_2$, and $\mathbf{e}_1\mathbf{e}_2$, each such cochain must be a cocycle because $C^2(\partial\Delta_2^\omega; \mathbb{F})$ is trivial (hence $\delta^1 = 0$). Also, a cochain f is a coboundary if and only if there is some 0-dimensional cochain g such that

$$f(\mathbf{e}_i\mathbf{e}_j) \;=\; g(\mathbf{e}_i) \;-\; g(\mathbf{e}_j)$$

for all i and j such taht $0 \leq i < j \leq 2$.

Now consider the cochain f with $f(\mathbf{e}_0\mathbf{e}_1) = f(\mathbf{e}_0\mathbf{e}_2) = f(\mathbf{e}_1\mathbf{e}_2) = 1$. We claim that f cannot be a coboundary. If it were, then as above we could find integers $x_i = g(\mathbf{v}_i)$ such that

$$x_1 - x_0 \quad = \quad x_2 - x_0 \quad = \quad x_2 - x_1 \quad = \quad 1 \; .$$

This is a system of three linear equations in three unknowns, but it has no solutions. The nonexistence of solutions means that f cannot possibly be a coboundary. Similar considerations show that if k is an integer which is prime to the characteristic of $\mathbb{F}$ (in the characteristic zero case this means $k \neq 0$), then $k \cdot f$ is a cocycle which is not a coboundary.

By the previous results on cohomology isomorphisms, it follows that the singular cohomology $H^1(S^1;\mathbb{F})$ and simplicial cohomology $H^1(\partial\Delta_2;\mathbb{F})$ must also be nonzero.■

RELATIVE CUP PRODUCTS. If $(P, \mathbf{K})$ is a simplicial complex and we are given two subcomplexes $(Q_i, \mathbf{L}_i)$ for $i = 1, 2$, then one can define relative cup products on the simplicial cochain level

$$C^i(\mathbf{K}, \mathbf{L}_1; \mathbb{D}) \times C^j(\mathbf{K}, \mathbf{L}_2; \mathbb{D}) \;\longrightarrow\; C^{i+j}(\mathbf{K}, \mathbf{L}_1 \cup \mathbf{L}_2; \mathbb{D})$$

in much the same way that one defines such products of singular cochains, and once again these products pass to bilinear maps of cohomology groups

$$H^i(\mathbf{K}, \mathbf{L}_1; \mathbb{D}) \times H^j(\mathbf{K}, \mathbf{L}_2; \mathbb{D}) \;\longrightarrow\; H^{i+j}(\mathbf{K}, \mathbf{L}_1 \cup \mathbf{L}_2; \mathbb{D}) \; .$$

Specifically, if $\mathbf{L}_1$ and $\mathbf{L}_2$ are both subcomplexes of $\mathbf{K}$ and we are given cochains

$$f : C^i(\mathbf{K}, \mathbf{L}_1; \mathbb{D}) \to \mathbb{D} \; , \quad g : C^j(\mathbf{K}, \mathbf{L}_2; \mathbb{D})$$

then the cochain level cup product

$$C^i(\mathbf{K}, \mathbf{L}_1; \mathbb{D}) \times C^j(\mathbf{K}, \mathbf{L}_2; \mathbb{D}) \;\longrightarrow\; C^{i+j}(\mathbf{K}, \mathbf{L}_1 \cup \mathbf{L}_2; \mathbb{D})$$

sends (f, g) to the cochain $f \cup g$ whose value on a simplex generator T of $C_{i+j}(\mathbf{K})$ is the product of f evaluated on the front i-face of T and g evaluated on the back j-face of T. Since f and g are cochains which vanish on $C_i(\mathbf{L}_1)$ and $C_j(\mathbf{L}_2)$ respectively, it follows that $f \cup g$ vanishes on $C_{i+j}(\mathbf{L}_1 \cup \mathbf{L}_2)$ and hence defines a relative cochain. One can then reason exactly in the singular case to show that this cochain level cup product passes to a cup product in simplicial cohomology, and once again this refined cup product has analogs of all the properties one has in the singular case.

COMPATIBILITY OF THE SINGULAR AND SIMPLICIAL CUP PRODUCTS. Clearly it would be very useful to know that the singular and simplicial cup products correspond under the standard isomorphism from singular to simplicial cohomology. This is slightly less trivial than one might initially expect, for the relative cup product in singular cohomology is defined for pairs (X, A) such that A is open in X and the corresponding product in simplicial cohomology is defined for pairs (X, A) such that A is a closed subset of X. We shall need the following result in order to prove compatibility:

> *Suppose that $(P, \mathbf{K})$ is a simplicial complex and $(Q, \mathbf{L})$ is a $(Q_1, \mathbf{L}_1)$ and $(Q_2, \mathbf{L}_2)$ are subcomplex. Then there is an open neighborhood W of Q in P such that Q is a deformation retract of W.*

There is a proof of this result in Section II.9 of Eilenberg and Steenrod (and there are also proofs in many other algebraic topology texts).■

Now suppose that $(P, \mathbf{K})$ is a simplicial complex and that $(Q_1, \mathbf{L}_1)$ and $(Q_2, \mathbf{L}_2)$ are subcomplexes. Let W_1 and W_2 be open subsets of P such that Q_i is a deformation retract of W_i for $i = 1, 2$. Then a Five Lemma argument implies that the restriction mappings $H^*(P, W_i) \to H^*(P, Q_i)$ are isomorphisms, and the following result relates the singular and simplicial cup products:

THEOREM 7. *In the setting of the preceding paragraph, we have the following commutative diagram*

$$\begin{array}{ccc} H^s(P, W_1; \mathbb{D}) \times H^t(P, W_2; \mathbb{D}) & \xrightarrow{\cup} & H^{s+t}(P, W_1 \cup W_2; \mathbb{D}) \\ \big\downarrow j_1^* \times j_2^* & & \big\downarrow j^* \\ H^s(P, Q_1; \mathbb{D}) \times H^t(P, Q_2; \mathbb{D}) & & H^{s+t}(P, Q_1 \cup Q_2; \mathbb{D}) \\ \big\downarrow \theta^* \times \theta^* & & \big\downarrow \theta^* \\ H^s(\mathbf{K}, \mathbf{L}_1; \mathbb{D}) \times H^t(\mathbf{K}, \mathbf{L}_2; \mathbb{D}) & \xrightarrow{\cup} & H^{s+t}(\mathbf{K}, \mathbf{L}_1 \cup \mathbf{L}_2; \mathbb{D}) \end{array}$$

in which the terms are given as follows and have the specified properties:

(i) *The horizontal arrows in the top and bottom row denote the singular and simplicial cup products.*

(ii) *The the mappings j_1^*, j_2^*, and j^*, are (restriction) maps induced by the appropriate inclusions of pairs, and the first two maps (and hence also their product) are isomorphisms.*

(iii) *The maps θ^* are the usual natural isomorphisms from singular to simplicial cohomology.*

It follows that the horizontal arrow in the first column is an isomorphism, and in fact a more precise application of the results from Eilenberg and Steenrod implies that we can choose the neighborhoods W_i so that the horizontal arrow in the second column is also an isomorphism, although this is not needed for many applications. Frequently we shall abuse language and say that the bottom line is the relative cup product in singular cohomology.

Method of proof. The proof follows immediately from the definitions of the various morphisms in the diagram (verify this!).■

IV.2 : A weak Universal Coefficient Theorem

(Hatcher, § 3.1)

We have already asserted the q-dimensional cohomology of a space is the dual space of the q-dimensional homology if we take coefficients in a field. However, our basic definition is somewhat different from this, so the next step is to verify the assertion at the beginning of this unit. Hatcher formulates and proves more general results (for example, see Theorem 3.2 on page 195). In this course we do not have enough time to develop the homological algebra necessary to prove such a result, and in any case the results for fields are strong enough to yield some important insights; one slogan might be that our setting only requires linear algebra and not the full force of homological algebra. However, if one goes deeper into the subject then it is necessary to work in the category of modules over arbitrary principal ideal domains.

As usual let $\mathbb{D}$ be a commutative ring with unit, let C_* be a chain complex of $\mathbb{D}$-modules, and define an associated cochain complex by $C^q = \mathrm{Hom}_{\mathbb{D}}(C_q, \mathbb{D})$, with a coboundary map $d^q = \mathrm{Hom}(d_{q+1}, \mathbb{D})$ analogous to the construction for singular cochains. Then evaluation defines a bilinear map $C^q \times C_q \to \mathbb{D}$ which is called the *Kronecker index pairing* and its value at $f \in C^q$ and $x \in C_q$ is usually written as $\langle f, x \rangle$.

LEMMA 1. *Suppose that $f, f' \in C^q$ are cocycles and $x, x' \in C_q$ are cycles such that $f - f' = \delta a$ and $x - x' = db$. Then $\langle f, x \rangle = \langle f', x' \rangle$.*

Proof. For an arbitrary cochain g and chain y it follows immediately that $\langle \delta g, y \rangle = \langle g, dy \rangle$. Therefore we have

$$\langle f, x - x' \rangle \quad = \quad \langle f, db \rangle \quad = \quad \langle \delta f, b \rangle \;\; \langle 0, b \rangle \quad = \quad 0$$

and similarly

$$\langle f - f', x' \rangle \quad = \quad \langle \delta a, x' \rangle \quad = \quad \langle a, dx' \rangle \;\; \langle a, 0 \rangle \quad = \quad 0$$

which combine to show that $\langle f, x \rangle = \langle f', x' \rangle$.■

COROLLARY 2. *The chain/cochain level Kronecker index pairing passes to a well-defined bilinear pairing from $H^q(C) \times H_q(C)$ to $\mathbb{D}$.*■

Manipulations with dual vector spaces

We now assume that $\mathbb{F}$ is a field. If V is a vector space over $\mathbb{F}$ and U is a subspace of V, then we have a short exact sequence of vector spaces

$$0 \;\to\; U \;\to\; V \;\to\; V/U \;\to\; 0$$

and applying the dual space functor we obtain the following short exact sequence of dual spaces:

$$0 \;\to\; (V/U)^* \;\to\; V^* \;\to\; U^* \;\to\; 0$$

The image of the map from $(V/U)^*$ to V^* is the **annihilator** of U, which consists of all linear functionals which vanish on U and will be denoted by $U^\dagger$.

Suppose now that V_1 and V_2 are vector spaces over $\mathbb{F}$ and $T : V_1 \to V_2$ is a linear transformation. Then we can factor T into a composite

$$V_1 \to J_1 \cong J_2 \subset V_2$$

where J_1 is the quotient of V_1 by the kernel of T, the map from J_1 to J_2 is an isomorphism, and J_2 is the image of T. There is also a corresponding factorization for the induced map of dual spaces

$$V_2^* \to J_2^* \cong J_1^* \subset V_1^*$$

These factorizations will be useful in proving the following abstract version of a key result in linear algebra:

PROPOSITION 3. *In the notation above, let $T^* : V_2 \to V_1^*$ be the associated map of dual spaces. Then we have* $(\mathrm{Kernel}\ T)^\dagger = \mathrm{Image\ T}^* \subset V_1^*$ *and* $(\mathrm{Image}\ T)^\dagger = \mathrm{Kernel}\ T^* \subset V_2^*$.

Proof. By our previous observations we know that (Kernel $T)^{\dagger}$ corresponds to $J_1^* = J_2^*$, and since J_2 is the image of T, we have the asserted relationship. Similarly, we know that $(\text{Image}^T)^{\dagger}$ corresponds to $(V_2/J_2)^*$, and one can check directly that this corresponds to all linear functionals f on V_2 such that $0 = f \circ T = T^*(f)$.∎

We now have enough machinery to derive the relationship between homology and cohomology over a field.

PROPOSITION 4. *Let C_* be a chain complex over a field $\mathbb{F}$, and let C^* be the dual cochain complex. Then for each q there is a natural isomorphism from $H^q(C)$ to $H_q(C)^*$.*

Proof. We shall focus on verifying the assertion about the isomorphism first. By definition we know that

$$H^q \cong (\text{Kernel } \delta^q)/(\text{Image } \delta^{q-1}) .$$

Using the relationship $\delta = d^*$ we may rewrite the right hand side in the form

$$(\text{Image } d_{q+1})^{\dagger}/(\text{Kernel } d_q)^{\dagger}$$

and conclude by noting that the latter subquotient of C_q^* corresponds to

$$H_q^* \cong \Big((\text{Kernel } d_q)/(\text{Image } d_{q+1})\Big)^* .$$

Under these correspondences and the defining isomorphism

$$H_q \cong \Big((\text{Kernel } d_q)/(\text{Image } d_{q+1})\Big)$$

all the standard pairings which evaluate linear functionals at vectors are preserved. In particular, this means that the isomorphism is given by the pairing described in Corollary 2. Now this pairing is natural by construction, and therefore our isomorphism is also natural.∎

Only a little more work is needed to derive the description of singular cohomology that we want.

COROLLARY 5. *If (X, A) is a topological space and $\mathbb{F}$ is a field, then for each q there is a natural isomorphism from $H^q(X, A; \mathbb{F})$ to the dual space $H_q(X, A; \mathbb{F})^*$.*

Proof. At this point all we need to do is describe a natural isomorphism

$$S^*(X, A; \mathbb{F}) \cong \text{Hom}(S_*(X, A), \mathbb{F}) \longrightarrow \text{Hom}_{\mathbb{F}}(S_*(X, A) \otimes \mathbb{F}, \mathbb{F})$$

because the latter is the cochain complex to which Proposition 4 applies. However, the isomorphism in question is given directly by the universal properties of the tensor product construction sending the chain groups $S_q(X, A)$ to $S_q(X, A) \otimes \mathbb{F}$; in other words, there is a 1–1 correspondence between abelian group homomorphisms from $S_q(X, A)$ to $\mathbb{F}$ and $\mathbb{F}$-linear maps from $S_q(X, A) \otimes \mathbb{F}$ to $\mathbb{F}$.∎

If (X, A) is a pair of topological spaces, then similar considerations show that under this isomorphism the connecting morphism in cohomology

$$\delta^* : H^p(A; \mathbb{F}) \longrightarrow H^{p+1}(X, A; \mathbb{F})$$

corresponds to the map $\text{Hom}_{\mathbb{F}}(\partial, \mathbb{F})$, where $\partial : H_{p+1}(X, ; \mathbb{F}) \to H_p(A; \mathbb{F})$ is the connecting morphism in homology. This reflects the fact that chain complex boundaries and cochain complex coboundaries are adjoint to each other with respect to the Kronecker index pairing; details of the verification are left to the reader$^{(\star)}$.

IV.3 : Künneth formulas

(Hatcher, §§ 3.2, 3.B)

One obvious point about the preceding discussion is that we have not yet produced examples for which the cup product of two positive-dimensional cohomology classes is nontrivial. Our next order of business is to find classes of examples with this property. The first step is to prove purely algebraic versions of the results we want.

Algebraic cross products

The proof of the topological result in the preceding paragraph depends on finding a suitable chain complex for computing the homology of a product space $X \times Y$; more precisely, we want this to be an algebraic construction on the singular chain complexes of X and Y which is somehow an algebraic product of $S_*(X)$ and $S_*(Y)$. The correct model is given by a tensor product construction.

Definition. Let (A_*, d^A_*) and (B_*, d^B_*) be chain complexes over a principal ideal domain $\mathbb{D}$ such that the chain groups in negative dimensions are zero. Then the tensor product $(A_*, d^A_*) \otimes_{\mathbb{D}} (B_*, d^B_*)$ has chain groups

$$(A \otimes B)_n \quad = \quad \bigoplus_{p=0}^{n} A_p \otimes B_{n-p}$$

and the differential is given on $A_p \otimes B_q$ by the formula

$$d^{A\otimes B}(x \otimes y) \quad = \quad d^A(x) \otimes y \; + \; (-1)^p x \otimes d^B(y) \; .$$

The sign is needed to ensure that $d^{A\otimes B} \circ d^{A\otimes B} = 0$ so that we actually get a chain complex; proving this algebraic identity is a fairly straightforward (and not too messy) exercise. It is also fairly straightforward to verify that this construction is covariantly functorial in A and B.$^{(\star)}$

If we are simply given graded modules A_* and B_* (which may be viewed as chain complexes with zero differentials), then the preceding also yields a definition of the tensor product $A_* \otimes B_*$.

We shall need the following elementary observation, whose proof is left to the reader$^{(\star)}$.

PROPOSITION 1. *If B_* is a graded module which is free in all gradings (each B_k is free) and we are given a short exact sequence $0 \to K_* \to A_* \to C_* \to 0$ of graded modules, then the tensor product*

$$0 \;\longrightarrow\; K_* \otimes_{\mathbb{D}} B_* \;\longrightarrow\; A_* \otimes_{\mathbb{D}} B_* \;\longrightarrow\; C_* \otimes_{\mathbb{D}} B_* \to 0$$

is also a short exact sequence of graded modules. A similar conclusion holds for all B_* provided each of the graded modules K_*, A_*, C_* is free.■

One important consequence of the definition for tensor products of chain complexes is the following method of constructing classes in $H_*(A \otimes B)$ from classes in $H_*(A)$ and $H_*(B)$.

PROPOSITION 2. *In the setting above, there are bilinear mappings*

$$\times : H_p(A;\mathbb{D}) \ \otimes \ H_q(B;\mathbb{D}) \ \longrightarrow \ H_{p+q}(A\otimes B;\mathbb{D})$$

with the following property: If $x \in A_p$ is a cycle representing u and $y \in B_q$ is a cycle representing v, then $x \otimes y$ is a cycle representing $u \times v$.

This construction is called the *external homology cross product.*

Proof. We shall only sketch the main steps and leave the details to the reader$^{(\star)}$. First of all, if x and y are cycles, then the definitions imply that $x \otimes y$ is also a cycle, and if $x = dw$ or $y = dz$ then $x \otimes y$ is a boundary. Bilinearity follows from the definition, and this plus the preceding sentence imply that the bilinear map is well defined.■

Our next result states that these products are maximally nontrivial if $\mathbb{D}$ is a field.

THEOREM 3. (The algebraic Künneth Theorem) *If $\mathbb{F}$ is a field, then the external homology cross product defines an isomorphism*

$$\bigoplus_{p=0}^{n} H_p(A;\mathbb{F}) \otimes H_{n-p}(B;\mathbb{F}) \quad \longrightarrow \quad H_n(A \otimes B;\mathbb{F})$$

for all $n \geq 0$.

Proof. In the argument below, all tensor products are taken over the field $\mathbb{F}$.

For each integer k let $\mathfrak{z}(A_q) \subset A_q$ be the subspace of cycles; if we define a chain complex structure on $\mathfrak{z}(A_*)$ by setting all boundary homomorphisms equal to zero, then $\mathfrak{z}(A_*)$ is a chain subcomplex of A_*, and the quotient complex

$$\mathfrak{q}(A_*) \quad = \quad A_*/\mathfrak{z}(A_*)$$

also has a trivial differential because $\mathfrak{q}(A_k) \cong d[A_k] \subset A_{k-1}$ and $d \circ d = 0$.

Since we are working over a field $\mathbb{F}$, all modules are free, and hence if we apply Proposition 3 to the previous short exact sequence we obtain a short exact sequence of chain complexes

$$0 \ \longrightarrow \ \mathfrak{z}(A_*) \otimes_{\mathbb{F}} B_* \ \longrightarrow \ A_* \otimes_{\mathbb{F}} B_* \ \longrightarrow \ \mathfrak{q}(A_*) \otimes_{\mathbb{F}} B_* \ \longrightarrow \ 0$$

which of course has an associated long exact homology sequence. Since the differentials in $\mathfrak{z}(A_*)$ and $\mathfrak{q}(A_*)$ are trivial, this long exact sequence has the form

$$\cdots \ \ [\mathfrak{z}(A)_* \otimes H_*(B)]_k \ \ \rightarrow \ \ H_k((A \otimes B) \ \ \rightarrow \ \ [\mathfrak{q}(A)_* \otimes H_*(B)]_k \ \ \rightarrow \ \ [\mathfrak{z}(A)_* \otimes H_*(B)]_{k-1} \ \ \cdots$$

and the definitions of the connecting homomorphisms imply that the right hand arrow ∂_k is given by $\widetilde{d}_* \otimes \mathrm{id}\,[H_*(B)]$, where

$$\widetilde{d}_m : \mathfrak{q}(A)_m \quad \longrightarrow \quad \mathfrak{z}(A)_{m-1}$$

is the composite

$$\widetilde{d}_m : \mathfrak{q}(A)_m \quad \cong \quad d_m[A_m] \quad \subset \quad \mathfrak{z}(A)_{m-1} \ .$$

This implies that the map $\widetilde{d}_m$ is injective, and since the tensor product functor over a field preserves short exact sequences it follows that the connecting homomorphisms ∂_k are also injective. Therefore the maps

$$H_k((A \otimes B) \quad \rightarrow \quad [\mathfrak{q}(A)_* \otimes H_*(B)]_k$$

are zero, so by exactness it follows that $H_k(A \otimes B)$ is isomorphic to the quotient

$$[\mathfrak{z}(A)_* \otimes H_*(B)]_k \ / \ [\mathfrak{q}(A)_* \otimes H_*(B)]_k \quad \cong \quad [H_*(A)_* \otimes H_*(B)]_k$$

and a check of the definitions shows that the isomorphism is induced by the homology cross product.■

There is also a **dual cross product in cohomology**. If $g : A_p \to \mathbb{F}$ and $h : B_q \to \mathbb{F}$ are cochains, then we define a cross product cochain $g \times h : [A \otimes B]_{p+q} \to \mathbb{F}$ such that the restriction to $A_r \otimes B_{p+q-r}$ is zero if $r \neq p$ and the restriction to $A_p \otimes B_q$ satisfies the identity

$$g \times h(x \otimes y) \quad = \quad g(x) \cdot h(y) \ , \qquad (x \in A_p \ , \quad y \in B_q) \ .$$

The coboundary of $g \times h$ is given by the following identity:

LEMMA 4. *In the setting above we have*

$$\delta(g \times h) \quad = \quad \delta g \times h \ + \ (-1)^p \, g \times \delta h \ .$$

In particular, if g and h are cocycles then so is $g \times h$, and if in addition one of g and h is a coboundary then so is $g \times h$, so that the cross product passes to a bilinear mapping from $H^p(A) \otimes H^q(B) \to H^{p+q}(A \otimes B)$.

Sketch of proof. By definition we have

$$\delta(g \times h) \quad = \quad (g \times h)^\circ d \quad = \quad (g \times h)^\circ \left(d^A \otimes \mathrm{id} \ + \ (-1)^p \mathrm{id} \otimes d^B \right)$$

and if we apply the right hand expression to a typical generator $z \otimes w \in A_m \otimes B_{p+q-m-1}$ we see that the value equals the value of the right hand side in the displayed expression of the lemma. The second sentence in the lemma follows by adding the conditions $\delta g = \delta h = 0$ for the first assertion, and adding one of the additional conditions $g = \delta g'$ or $h = \delta h'$ in the second. The third sentence follows immediately from these and the fact that the cochain cross product is bilinear.■

It is now very easy to show that the cross product of two nontrivial cohomology classes is nonzero.

COROLLARY 5. *If $\alpha \in H^p(A)$ and $\beta \in H^q(B)$ are nonzero, then so is $\alpha \times \beta$.*

Proof. By the Weak Universal Coefficient Theorem there are homology classes $u \in H_p(A)$ and $v \in H_q(B)$ such that the Kronecker indices $\alpha(u)$ and $\beta(v)$ are nonzero elements of $\mathbb{F}$. Since the Kronecker index of the cohomology cross product satisfies

$$\langle \alpha \times \beta, u \times v \rangle \quad = \quad \alpha(u) \cdot \beta(v)$$

and the right hand side is nonzero (it is a product of two nonzero elements in $\mathbb{F}$), it follows that $\alpha \times \beta$ is also nonzero.■

Topological cross products

We can define the cross product of two singular cochains by a variant of the cup product definition. If $\mathbb{D}$ is a commutative ring with unit and we are given two singular cochains $f : S_p(X, \mathbb{D}) \to \mathbb{D}$ and $g : S_q(Y, \mathbb{D}) \to \mathbb{D}$, then their cross product

$$f \times g : S_{p+q}(X \times Y; \mathbb{D}) \ \longrightarrow \ \mathbb{D}$$

is defined on a singular simplex $T = (T_X, T_Y) : \Delta_{p+q} \to X \times Y$ by the formula

$$f \times g(T) \quad = \quad f\,(\,\mathbf{Front}_p(T_X)\,) \cdot g\,(\,\mathbf{Back}_q(T_Y)\,) \ .$$

The usual bilinearity and associativity properties follow directly from the definition (details are left to the reader). We also have the following identities showing that each of the cup and cross products can be easily described in terms of the other:

PROPOSITION 6. *In the setting above we have the following identities, whose verifications are left to the reader:*

(i) *If $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are coordinate projections with associated singular cochain homomorphisms $\pi_X^\#$ and $\pi_Y^\#$, then $f \times g = \pi_X^\#(f) \cup \pi_Y^\#(g)$.*

(ii) *If $X = Y$ and $\Delta_X : X \to X \times X$ is the diagonal map, then $f \cup g = \Delta_X^\#(f \times g)$.*∎

The singular cohomology cross product also satisfies analogs of the basic properties for cohomology products in Lemma 4;

LEMMA 7. *In the setting above we have*

$$\delta(f \times g) \quad = \quad \delta f \times g \ + \ (-1)^p \, f \times \delta g \ .$$

In particular, if f and g are cocycles then so is $f \times g$, and if in addition one of f and g is a coboundary then so is $f \times g$, so that the cross product passes to a bilinear mapping from $H^p(X;\mathbb{D}) \otimes H^q(Y;\mathbb{D}) \to H^{p+q}(X \times Y;\mathbb{D})$.∎

The Topological Künneth Theorem

At this point we need a result relating the singular homology relating the singular homology of $X \times Y$ to the singular homology of the factors X and Y; to shorten the discussion, we restrict ourselves to field coefficients in these notes. Some of the earliest general results of this type were due to H. Künneth in the early 1920s, and in singular homology this relationship follows from a general method of **acyclic models** due to Eilenberg and J. A. Zilber. We shall not formulate this method abstractly, but the reader may be able to see the general pattern emerge.

THEOREM 8. (Eilenberg-Zilber) *If $\mathbb{D}$ is a principal ideal domain and X and Y are topological spaces, then there are functorial chain homotopy equivalences*

$$\psi_{X,Y} : S_*(X \times Y;\mathbb{D}) \ \to \ S_*(X;\mathbb{D}) \otimes_{\mathbb{D}} S_*(Y;\mathbb{D}) \ , \varphi_{X,Y} : S_*(X;\mathbb{D}) \otimes_{\mathbb{D}} S_*(Y;\mathbb{D}) \ \to \ S_*(X \times Y;\mathbb{D})$$

with the following properties:

(i) *The composites $\varphi \circ \psi$ and $\psi \circ \varphi$ are naturally chain homotopic to the identity.*

(ii) *In degree 0 the map ψ takes a singular 0-simplex $T = (T_X, T_Y)$ to $T_X \otimes T_Y$, and φ is inverse to ψ.*

We shall describe an explicit choice for $\psi_{X,Y}$ known as the *Alexander-Whitney map*.

Before proving Theorem 8, we shall list some of its consequences:

THEOREM 9. (Classical Künneth Formula for field coefficients) *Let X and Y be topological spaces, and let $\mathbb{F}$ be a field. Then the composite of the homology cross product and induced mapping ψ_* in homology defines isomorphisms of singular homology groups*

$$\bigoplus_{p=0}^{n} H_p(X;\mathbb{F}) \otimes_{\mathbb{F}} H_{n-p}(Y;\mathbb{F}) \quad \longrightarrow \quad H_n(X \times Y;\mathbb{F})$$

for all $n \geq 0$.

This result follows directly from Theorem 8 and the Algebraic Künneth Formula (Theorem 3).■

THEOREM 10. (Cohomological Künneth Formula) *Let* X *and* Y *be topological spaces, let* $\mathbb{F}$ *be a field, and assume that the homology groups* $H^p(X;\mathbb{F})$ *and* $H^q(Y;\mathbb{F})$ *are finite for all* $p.q \geq 0$. *Then the cohomology cross product map defines isomorphisms of singular homology groups*

$$\bigoplus_{p=0}^{n} H^p(X;\mathbb{F}) \otimes_{\mathbb{F}} H^{n-p}(Y;\mathbb{F}) \quad \longrightarrow \quad H^n(X \times Y;\mathbb{F})$$

for all $n \geq 0$.

Sketch of the proof that Theorem 9 implies Theorem 10. This is a consequence of the following observations:

(1) The Universal Coefficient isomorphism from $H^k(W;\mathbb{F})$ to the dual space of $H^k(W;\mathbb{F})$, where $W = X$ or Y and $k \geq 0$.

(2) The natural isomorphism $(V_1 \oplus V_2)^* \cong V_1^* \oplus V_2^*$, where V^* denotes the dual space and V_1 and V_2 are vector spaces over $\mathbb{F}$.

(3) The natural isomorphism $(V_1 \otimes_{\mathbb{F}} V_2)^* \cong V_1^* \otimes_{\mathbb{F}} V_2^*$, where V^* denotes the dual space and V_1 and V_2 are **finite dimensional** vector spaces over $\mathbb{F}$ (in fact, the conclusion of the theorem is generally false if the finite dimensionality conditions do not hold, but there still is a natural monomorphism from $V_1^* \otimes_{\mathbb{F}} V_2^*$ to $(V_1 \otimes_{\mathbb{F}} V_2)^*$).

Note that under these isomorphisms the homology and cohomology cross products correspond; namely, if $f_i \in V_i^*$ and $x_i \in V_i$ then $f_1 \otimes f_2(x_1 \otimes x_2) = f_1(x_1) \cdot f_2(x_2)$.■

The first step in proving Theorem 7 is to consider the special case where $X = Y = \Delta_n$ for some n.

LEMMA 11. *Let* $\mathbb{D}$ *be a commutative ring with unit. If* $p, q \geq 0$ *and an augmentation is defined on* $S_*(\Delta_n;\mathbb{D}) \otimes S_*(\Delta_n;\mathbb{D})$ *using the multiplication and tensor product maps*

$$S_0(\Delta_n : \mathbb{D}) \otimes_{\mathbb{D}} S_0(\Delta_n : \mathbb{D}) \ \rightarrow \ \mathbb{D} \otimes \mathbb{D} \ \rightarrow \ \mathbb{D}$$

then $S_*(\Delta_p;\mathbb{D}) \otimes_{\mathbb{D}} S_*(\Delta_q;\mathbb{D})$ *is acyclic.*

Proof of Lemma 11. Let $C_*(\mathbb{D})$ be the ordered simplicial chain complex $C_*(\{\mathbf{e}\}_0^\omega;\mathbb{D})$, where $\mathbf{e}\}_0$ is a standard vertex of Δ_n, let $\eta : C_*(\mathbb{D}) \to S_*(\Delta_p;\mathbb{D})$ be the augmentation-preserving inclusion determined by viewing the generator of $C_0(\mathbb{D})$ as the singular 0-simplex sending the unique point in Δ_0 to the vertex $\mathbf{e}\}_0 \in \Delta_p$, and note that the augmentation map ε on $S_*(\Delta_p)$ can be viewed as a chain map from the latter to $C_*(\mathbb{D})$. Then the proof of homotopy invariance for singular homology implies that $\eta[C_*(\mathbb{D})]$ is a chain homotopy deformation retract of $S_*(\Delta_p;\mathbb{D})$. We can then construct the tensor product of a contracting chain homotopy with the identity on $S_*(\Delta_q;\mathbb{D})$, and it follows immediately that $S_*(\Delta_q;\mathbb{D}) \cong C_*(\mathbb{D}) \otimes \otimes_{\mathbb{D}} S_*(\Delta_q;\mathbb{D})$ is a chain deformation retract of $S_*(\Delta_p;\mathbb{D}) \otimes_{\mathbb{D}} S_*(\Delta_q;\mathbb{D})$. Since the smaller chain complex is acyclic, it follows that the larger chain complex is also acyclic.■

SIMPLICIAL ANALOGS OF LEMMA 11. Similar results hold for the ordered and unordered simplicial chain complexes of $\Delta - n$. The proofs are straightforward adaptations of the proof in the singular case and are left to the reader.■

Proof of Theorem 8. The Alexander-Whitney map $\psi_{X,Y}$ is just a formalization of earlier constructions, Specifically, if $T = (T_X, T_Y) : \Delta_n : X \times Y$ is a singular simplex given by the coordinate projections T_X and T_Y, then

$$\psi_{X,Y}(T) \quad = \quad \sum_{p=0}^{n} \mathbf{Front}_p(T_X) \otimes \mathbf{Back}_{n-p}(T_Y) \ .$$

It is a routine exercise to check that this construction defines a natural chain map$^{(\star)}$.

The idea behind constructing φ and the chain homotopies is to look at universal examples and extend to the general case by naturality. The chain groups

$$[S_*(X;\mathbb{D}) \otimes S_*(Y;\mathbb{D})]_n$$

are free modules, and explicit free generators are given by all objects of the form $F_X \otimes B_Y$, where $F_X : \Delta_p \to X$ and $B_Y : \Delta_{n-p} \to Y$ are singular simplices and $0 \leq p \leq n$. We shall define φ on such objects recursively with respect to n. The stated conditions define φ in degree 0. Once we are given φ in degrees $\leq n-1$, we shall define φ first on the universal class

$$\mathrm{id}[\Delta_p] \otimes \mathrm{id}[\Delta_{n-p}] \in S_p(\Delta_p;\mathbb{D}) \otimes_{\mathbb{D}} S_{n-p}(\Delta_{n-p};\mathbb{D})$$

and then we shall define $\varphi(F_X \otimes B_Y)$ by the naturality condition

$$\varphi(F_X \otimes B_Y) \quad = \quad F_{X\#}\,(\mathrm{id}[\Delta_p]) \otimes B_{Y\#}\,(\mathrm{id}[\Delta_{n-p}]) \ .$$

It is a straightforward exercise to verify that this construction defines a chain map in degree n, and this completes the inductive step$^{(\star)}$.

By the preceding discussion, the construction of φ reduces to finding a choice $W(p,q)$ for $\varphi(\mathrm{id}[\Delta_p] \otimes \mathrm{id}[\Delta_p])$ which satisfies the chain map condition

$$dW(p,q) \quad = \quad \varphi \circ d\,(\mathrm{id}[\Delta_p] \otimes \mathrm{id}[\Delta_p]) \ .$$

Since $S_*(\Delta_p \times \Delta_{n-p};\mathbb{D})$ is acyclic, such a class exists if and only if the right hand side is a cycle, so everything comes down to computing the boundary map on the right hand side. By the induction hypothesis, we know that φ is a chain map in degree $n-1$, and therefore the boundary of the right hand side is given by

$$d\,(\,\varphi \circ d\,(\mathrm{id}[\Delta_p] \otimes \mathrm{id}[\Delta_p])\,) \quad = \quad (\varphi \circ d) \circ d\,(\mathrm{id}[\Delta_p] \otimes \mathrm{id}[\Delta_p])$$

which vanishes because $d \circ d = 0$ as required. This completes the inductive step and the construction of φ.

The chain homotopies from $\varphi \circ \psi$ and $\psi \circ \varphi$ to the respective identity maps are also constructed using universal examples. We shall start by constructing the chain homotopy $\varphi \circ \psi \simeq \mathbf{id}$. Since $\varphi_0 \circ \psi_0$ is the identity map, we can take $D_0 = 0$. Assume that se have defined D_k for $k < n$; as before, we first define D_n on the diagonal map $\mathrm{Diag}_n : \Delta_n \to \Delta_n \times \Delta_n$ and then we extend by naturality. The classes

$$\theta_{n+1} \quad = \quad D_n\,(\mathrm{Diag}_n) \in S_{n+1}(\Delta_n \times \Delta_n)$$

are required to satisfy the identity

$$d\theta_n \ = \ \mathrm{Diag}_n \ - \ \varphi\psi\,(\mathrm{Diag}_n) \ - \ D_{n-1} \circ d_n\,(\mathrm{Diag}_n)$$

for all $n > 0$. Once again, everything reduces to showing that the right hand side is a cycle because $S_*(\Delta_n \times \Delta_n; \mathbb{D})$ is acyclic. Much as before, one can use the inductive hypothesis

$$\varphi_{n-1} \circ \psi_{n-1} \; - \; \text{identity} \;\; = \;\; D_{n-2} \circ d_{n-1} \; + \; d_n \circ D_{n-1}$$

and $d \circ d = 0$ to prove that $\text{Diag}_n \; - \; \varphi\psi\,(\text{Diag}_n) \; - \; D_{n-1} \circ d_n\,(\text{Diag}_n)$ is a cycle. As before, one extends by naturality, and it is another formal exercise to check that the construction defines a natural chain homotopy from $\varphi \circ \psi$ to the identity.

Similar considerations yield the chain homotopy $E : \psi \circ \varphi \simeq \mathbf{id}$. In this case we must use the free generators of $S_*(X; \mathbb{D}) \otimes S_*(Y; \mathbb{D})$ described above, and we need Lemma 10 for the fact that $S_*(\Delta_p; \mathbb{D}) \otimes_{\mathbb{D}} S_*(\Delta_q; \mathbb{D})$ is acyclic.■

The next result, which yields many examples of nontrivial cross products in singular homology and cohomology, is an immediate consequence of the results in this section.

COROLLARY 12. *Let X and Y be nonempty topological spaces, and let $\mathbb{F}$ be a field.*

(i) If $u \in H_p(X; \mathbb{F})$ and $v \in H_q(Y; \mathbb{F})$ are nonzero, then so is $u \times v$.

(ii) If $\alpha \in H^p(X; \mathbb{F})$ and $\beta \in H^q(Y; \mathbb{F})$ are nonzero, then so is $\alpha \times \beta$.■

Products in relative homology groups

We would also like to have a version of Corollary 11 for cross products in relative homology and cohomology. There are a few complications, but one can develop a reasonably good theory in this case. The first step is a generalization of the Eilenberg-Zilber Theorem. For the rest of this section we shall assume that all coefficients lie in some commutative ring with unit $\mathbb{D}$ which is suppressed from the notation.

THEOREM 13. *Suppose that (X, A) and (Y, B) are pairs of spaces such that A and B are open in X and Y respectively. Then there is a relative cross product on the cochain level*

$$S^p(X, A) \otimes S^q(Y, B) \;\longrightarrow\; S^{p+q}(X \times Y, A \times Y \cup X \times B)$$

which is compatible with the absolute cross product defined in this unit. This product satisfies analogs of the coboundary formulas in the absolute case and passes to a cohomology cross product which is also compatible with the previous construction when $A = B = \emptyset$. Furthermore, if the coefficients lie in a field $\mathbb{F}$ and all the cohomology groups $H^p(X, A)$ and $H^q(Y, B)$ are finite dimensional vector spaces, then the cross product defines an isomorphism from $H^*(X, A) \otimes H^*(Y, B)$ to $H^*(X \times Y, A \times Y \cup X \times B)$.

Proof. Let $\mathcal{U}$ be the open covering of $A \times Y \cup X \times B$ given by $\{A \times Y, X \times B\}$. Then one can check directly that the composite of the cochain level cross product from $S^p(X, A) \otimes S^q(Y, B)$ to $S^{p+q}(X \times Y)$ with the restriction mapping

$$S^{p+q}(X \times Y) \;\to\; S^{p+q}_{\mathcal{U}}(A \times Y \cup X \times B) \;\;=\;\; \text{Hom}_{\mathbb{D}}(S_{p+q}(A \times Y) + S_{p+q}(X \times B), \mathbb{D})$$

is trivial; in other words, if f is a cochain on $S_p(X)$ which vanishes on $S_p(A)$ and g is a cochain on $S_q(Y)$ which vanishes on $S_q(B)$, then $f \times g$ vanishes on $S_{p+q}(A \times Y) + S_{p+q}(X \times B) = S^{\mathcal{U}}_{p+q}(A \times Y \cup X \times B)$. The Leibniz Formula for the coboundary of a cross product is an immediate consequence of the construction and known results when $A = B = \emptyset$ (recall that the relative cochain groups

$S^*(Z,C)$ are contained in the absolute groups $S^*(Z)$ as the subgroups of cochains whose restrictions to $S_*(C)$ are zero). It follows that the cochain level cross product passes to cohomology mappings

$$H^p(X,A) \otimes H^q(Y,B) \;\to\; H^{p+q}\left(S^*(X \times Y)/S^*_{\mathcal{U}}(A \times Y \cup X \times B)\right)$$

and since the inclusion of the latter in $S_{p+q}(A \times Y \cup X \times B)$ is a chain homotopy equivalence by the proof of excision we get the desired map from $H^p(X,A) \otimes H^q(Y,B)$ to $H^{p+q}(X \times Y, A \times Y \cup X \times B)$. This completes the derivation of the construction, We must now prove the relative Künneth Formula in the final sentence of the theorem.

Consider first the case where $B = \emptyset$. Since the short exact sequences of singular chain complexes for the pair (X,A) is split in each degree (the standard free generators of $S_k(A)$ are a subset of the standard free generators for $S_k(X)$), we have the following commutative diagram in which the vertical maps are Alexander-Whitney maps and the rows are short exact sequences; all tensor products are taken with respect to the field $\mathbb{F}$.

$$\begin{array}{ccccccccc} 0 & \to & S_*(A \times Y) & \to & S_*(X \times Y) & \to & S_*(X \times Y, A \times Y) & \to & 0 \\ & & \downarrow \psi & & \downarrow \psi & & \downarrow \psi & & \\ 0 & \to & S_*(A) \otimes S_*(Y) & \to & S_*(X) \otimes S_*(Y) & \to & S_*(X,A) \otimes S_*(Y) & \to & 0 \end{array}$$

Since the vertical maps on the left and center induce isomorphisms in homology, it follows that the vertical map on the right also induces isomorphisms in homology; in fact, this part of the argument does not require A to be an open subset of X.

Now consider the following commutative diagram, in which the second and third rows are short exact sequences of chain complexes and the maps denoted by ψ are Alexander-Whitney maps:

$$\begin{array}{ccccc} S_*(X \times B, A \times B) & & & & \\ \cong & & & & \\ S^{\mathcal{U}}_*(A \times Y \cup X \times B, A \times Y) & & & & \\ \downarrow \subset & & & & \\ S_*(A \times Y \cup X \times B, A \times Y) & \to & S_*(X \times Y, A \times Y) & \to & S_*(X \times Y, A \times Y \cup X \times B) \\ \downarrow ? & & \downarrow \psi & & \downarrow \psi \\ S_*(X,A) \otimes S_*(B) & \to & S_*(X,A) \otimes S_*(Y) & \to & S_*(X,A) \otimes S_*(Y,B) \end{array}$$

The question mark represents the following **claim**: *There is a chain map ψ' from $S_*(A \times Y \cup X \times B, A \times Y)$ to $S_*(X,A) \otimes S_*(B)$ whose restriction to the subcomplex $S_*(x \times B, A \times B) \cong S^{\mathcal{U}}_*(A \times Y \cup X \times B, A \times Y)$ is the usual Alexander-Whitney map.* The existence of ψ' follows from the commutativity of the right hand square, for the latter implies that $\psi : S_*(X \times Y, A \times Y) \to S_*(X,A) \otimes S_*(Y)$ maps the kernel of the surjection from $S_*(X \times Y, A \times Y)$ to the kernel of the surjection from $S_*(X,A) \otimes S_*(Y)$. Furthermore, the commutativity of the diagram

$$\begin{array}{ccc} S_*(X \times B, A \times B) & \to & S_*(X \times Y, A \times Y) \\ \downarrow \psi & & \downarrow \psi \\ S_*(X,A) \otimes S_*(B) & \to & S_*(X,A) \otimes S_*(Y) \end{array}$$

implies that ψ' extends the Alexander-Whitney map ψ on $S_*(X, A) \otimes S_*(B)$.

By earlier discussion of special cases, which applies to the pair $(X \times B, A \times B)$ by interchanging the roles of the first and second factors, we know that the Alexander-Whitney map on $S_*(X, A) \otimes S_*(B)$ induces isomorphisms in singular homology, and by the proof of excision we know that the inclusion of $S_*(X \times B, A \times B)$ in $S_*(A \times Y \cup X \times B)$ induces isomorphisms in singular homology, and it follows immediately that the mapping ψ' also induces isomorphisms in singular homology. We have already shown that the Alexander-Whitney map from $S_*(X \times Y, A \times Y)$ induces an isomorphism in homology, and therefore the Five Lemma implies that the Alexander-Whitney map from $S_*(X \times Y, A \times Y \cup X \times B)$ also induces isomorphisms in homology.

If we now take coefficients in a field and assume all homology and cohomology groups are finite dimensional, then the weak Universal Coefficient Theorem implies that the dual cochain complex maps

$$S^*(X, A; \mathbb{F}) \otimes_{\mathbb{F}} S^*(Y, B; \mathbb{F}) \ \to S^*(X \times Y, A \times Y \cup X \times B; \mathbb{F})$$

induce isomorphisms in cohomology from $H^*(X, A; \mathbb{F}) \otimes_{\mathbb{F}} H^*(Y, B; \mathbb{F})$ to $H^*(X \times Y, A \times Y \cup X \times B; \mathbb{F})$.■

In particular, just as before we know that if the homology groups of (X, A) and (Y, B) are finite dimensional over $\mathbb{F}$ in each dimension, then the cross product of a nontrivial cohomology class $\alpha \in H^*(X, A; \mathbb{F})$ and a nontrivial cohomology class $\beta \in H^*(Y, B; \mathbb{F})$ will always be nontrivial.

CORRESPONDING RESULTS FOR CLOSED SUBSETS. Frequently we want versions of the preceding when A and B are closed subsets rather than open subsets. As in earlier discussions, analogous results hold if we assume that A and B are deformation retracts of open neighborhoods $A \subset U \subset X$ and $B \subset V \subset Y$ (details are left to the reader$^{(\star)}$ — the crucial point is that pairs like (X, A) and (X, U) have isomorphic homology), and in many (most?) situations of interest in algebraic and geometric topology this sort of condition is satisfied. For example, this is the case if X is a polyhedron and A is a subpolyhedron.

Cap products

Although the homology groups of a space do not have a ring structure, it turns out that the graded object $H_*(X, \mathbb{D})$ is a graded module over the cohomology ring if one multiply cohomology degrees by (-1).

Definition. Let X be a space, and let A_1 and A_2 be open subsets of X. The chain/cochain level cap product

$$\cap : S^p(X, A_1; \mathbb{D}) \otimes_{\mathbb{D}} S_n(X; \mathbb{D})/[S_n(A_1; \mathbb{D}) + S_n(A_2; \mathbb{D})] \quad \longrightarrow \quad S_{n-p}(X, A_2; \mathbb{D})$$

is defined as follows: Given $g : S_p(X, A_1) \to \mathbb{D}$ and a singular simplex $T : \Delta_n \to X$, the cochain $g \cap T$ is given by $g\big(\mathbf{Front}_p(T)\big) \cdot \mathbf{Back}_{n-p}(T)$. Strictly speaking this construction is defined on $S^p(X, A_1; \mathbb{D}) \otimes S_n(X; \mathbb{D})$, but it factors through the displayed quotient because $g \cap T$ is trival on all singular simplices in $S_n(A_1; \mathbb{D}) + S_n(A_2; \mathbb{D}) \subset S_n(X; \mathbb{D})$; if the image of T lies in A_2, then the image is trivial by the definition of $S_{n-p}(X, A_2; \mathbb{D})$, and if the image of T lies in A_1 then triviality follows because $f|S_p(X, A_1; \mathbb{D})$ is zero. If c is a p-chain in $S^p(X, A_1; \mathbb{D})$, then one has the usual sort of graded Leibniz rule for computing $d(g \cap c)$, and it follows that (1) $g \cap c$ is a cycle if g is a cocycle and c is a cycle, (2) $g \cap c$ is a boundary if either g is a coboundary or c is a boundary. Since A_1 and A_2 are open subsets of X, the proof of excision implies that the chain complex inclusion

$$S_*(A_1; \mathbb{D}) + S_*(A_2; \mathbb{D}) \quad \subset \quad S_*(A_1 \cup A_2; \mathbb{D})$$

induces isomorphisms in homology, it follows that the chain/cochain level cap product induces a map in homology/cohomology

$$\cap : H^p(X, A_1; \mathbb{D}) \otimes_{\mathbb{D}} H_n(X, A_1 \cup A_2; \mathbb{D}) \longrightarrow S_{n-p}(X, A_2; \mathbb{D}) \ .$$

The cap product map is $\mathbb{D}$-bilinear, and it also has the following formal properties:

PROPOSITION 14. *Let X be a space. Then the cap product has the following properties:*

(i) If $\varepsilon_X : S_0(X) \to \mathbb{D}$ is the augmentation and $[\varepsilon_X] \in H^0(X; \mathbb{D})$ is its cohomology class, then cap product with $[\varepsilon_X]$ induces the identity on $H_*(X; \mathbb{D})$.

(ii) The cap and cup product satisfy a mixed associative law: If $u \in H^q(X; \mathbb{D})$, $v \in H^p(X; \mathbb{D})$, and $z \in H_n(X; \mathbb{D})$, then $(u \cup v) \cap z = u \cap (v \cap z)$.

(iii) If $f : X \to Y$ is continuous with $u \in H^p(Y; \mathbb{D})$ and $z \in H_n(x; \mathbb{D})$, then $f_*(f^*u \cap z) = u \cap f_*(z)$.

In each case, one can verify that the corresponding identities hold at the chain/cochain level; details are left to the reader$^{(\star)}$.■

IV.4 : Grade-commutativity and examples

(Hatcher, §§ 3.2, 3.B)

DEFAULT HYPOTHESES. Unless explicitly stated otherwise, througout this section $\mathbb{D}$ will denote a commutative ring with unit, all chain complexes will be assumed to be modules over $\mathbb{D}$, and tensor products will be assumed to be given over $\mathbb{D}$.

It was fairly easy to prove that the cup product on cohomology is associative, and in fact this is also true on the cochain level. Furthermore, it was even easier to prove that the augmentation cocycle $\varepsilon : S_*(X) \to \mathbb{D}$ is a two sided identity in the absolute case (pairs where the subspace is empty). We shall now consider commutativity properties of cup products, both at the cohomology level and at the cochain level.

One can do direct calculations to show that the cup product is usually not commutative in the standard sense. For example, one can check this in the simplicial cohomology of complexes homeomorphic to $T^2 = S^1 \times S^1$. Our results contain both good news and bad news:

> Good news. On the cohomology level the cup product is **grade-commutative** in the sense that if $\alpha \in H^p(X, \mathbb{D})$ and $\beta \in H^q(Y, \mathbb{D})$, then $\beta \cup \alpha = (-1)^{pq} \alpha \cup \beta$.
>
> Bad news. On the cochain level the cup product is usually not even grade-commutative, although it is so up to a system of *higher chain homotopies* (however, we shall only show commutativity up to an ordinary chain homotopy).

In particular, it turns out that the Steenrod squares and reduced powers in Section 4.L of Hatcher are defined using such higher chain homotopies and in fact imply the impossibility of constructing a grade-commutative cup product on the cochain level for coefficients in a field $\mathbb{F}$ of finite characteristic. On a more positive note, relatively recent results of M. Mandell show that if X is reasonably nice — for example, if X is a polyhedron — then the homotopy type of X is determined by the singular chain complex together with the a suitably defined structure of higher chain homotopies for cup product commutativity. Here is the reference:

M. A. Mandell. *Cochains and homotopy type*, Publ. Math. Inst. Hautes Études Sci. **103** (2006), 213–246.

In contrast, if we work over a field $\mathbb{F}$ of characteristic zero, then it is possible to define cohomology groups using cochain constructions that are grade commutative (on the cochain level). There is a more extensive discussion of commutative cochains on pages 110–111 of the following book:

P. A. Griffiths and J. W. Morgan. Rational homotopy theory and differential forms, Progress in Mathematics Vol. 16. Birkhäuser, Boston, MA, 1981.

In the next unit we shall discuss some fundamental constructions which are closely related to the topics covered in this book.

Coalgebraic structures on simplicial chain complexes

Since homology and cohomology are essentially dual to each other, the existence of a product structure in the latter but not the former may seem puzzling. However, it turns out that one can resolve this by a more systematic approach to dualization. One can view a multiplicative structure on a (graded) algebraic object as a (grade-preserving0 homomorphism $\mu : A \otimes A \to A$; dually, one can define a COMULTIPLICATIVE structure as a homomorphism $\psi : A \to A \otimes A$; such a structure can also be called a **coproduct**. Every concept that is meaningful for an algebra or product has a natural dual concept which is meaningful for a coalgebra or coproduct. For example, just as an algebra is associative if and only if the diagram

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu\otimes 1} & A \otimes A \\ \downarrow 1\otimes\mu & & \downarrow \mu \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

commutes, we can say that an algebra is *coassociative* if the diagram

$$\begin{array}{ccc} A & \xrightarrow{\psi} & A \otimes A \\ \downarrow \psi & & \downarrow 1\otimes\psi \\ A \otimes A & \xrightarrow{\psi\otimes 1} & A \otimes A \otimes A \end{array}$$

commutes. All of this is intrinsically formal, but the next result shows that such structures actually arise in concrete situations.

PROPOSITION 1. *If X is a topological space, let Δ_X be the diagonal and let $\psi_{X,X}$ be the Alexander-Whitney map for $X \times X$. Then the chain map $\Psi_X = \psi_{X,X} \circ \Delta_{X\#}$ defines a coassociative comultiplication on $S_*(X)$. Furthermore, if $f : X \to Y$ is a continuous mapping, then the induced map of singular chain complexes is a morphism of coalgebras.*

This follows directly from the definition of the Alexander-Whitney map (details are left to the reader$^{(\star)}$).■

COROLLARY 2. *If our underlying commutative ring with unit is a field $\mathbb{F}$, then the chain level comultiplication induces a comultiplication in homology, and this comultiplication is functorial and coassociative.*■

The conceptual point of the proposition and corollary is that one can view the multiplicative structure in cohomology as the dual of the given comultiplicative structure in homology.

We can view a *two-sided identity* in an algebra as a homomorphism from $\mathbb{D} \to A$ such that the composites

$$A \cong \mathbb{D} \otimes A \to A \otimes A \to A \ , \qquad A \cong A \otimes \mathbb{D} \to A \otimes A \to A$$

are the identity mapping (in the graded case, we assume $\mathbb{D}$ is contained in degree zero). The dual notion is basically just an augmentation $A \to \mathbb{D}$, and of course it is supposed to satisfy the dual conditions that the composite mappings

$$A \to A \otimes A \to \mathbb{D} \otimes A \cong A \ , \qquad A \to A \otimes A \to A \otimes \mathbb{D} \cong A$$

are the identity. It is routine to verify that the standard augmentation maps on singular chain complexes have this property, so in fact the singular chain complex may be viewed as a functor from spaces to coassociative coalgebra chain complexes with augmentations.

Algebraic and topological twist maps

The preceding discussion indicates that grade-commutativity properties of cup products should be dual to *grade-cocommutativity* properties of the functorial comultiplication on singular chain complexes. At this point we need to introduce some algebra.

Definition. Suppose that A_* and B_* are chain complexes over $\mathbb{D}$. The transposition or twist isomorphism

$$\tau_{A,B} : A_* \otimes B_* \longrightarrow B_* \otimes A_*$$

is determined by the identity $\tau_{A,B}(a_p \otimes b_q) = (-1)^{pq}(b_q \otimes a_p)$, where $a_p \in A_p$ and $b_q \in B_q$.

It follows immediately that τ is a functorial chain map and $\tau_{B,A} \circ \tau_{A,B}$ is the identity$^{(\star)}$.

One motivation for this construction is the following result:

THEOREM 3. *Let X and Y be topological spaces, and let $T : X \times Y \to Y \times X$ be the map $T(x,y) = (y,x)$ which transposes coordinates. Then there is a commutative diagram up to natural chain homotopy*

$$\begin{array}{ccc} S_*(X \times Y) & \xrightarrow{\psi} & S_*(X) \otimes S_*(Y) \\ \big\downarrow T_\# & & \big\downarrow \tau \\ S_*(X \times Y) & \xrightarrow{\psi} & S_*(X) \otimes S_*(Y) \end{array}$$

in which the horizontal maps are Alexander-Whitney maps and τ is the algebraic transposition map on $S_*(X) \otimes S_*(Y)$.

Proof. The first things to observe is that all maps of chain complexes are augmentation preserving and the diagram commutes in degree zero. Assume inductively that the diagram commutes up to natural chain homotopy through dimension $n - 1 \geq 0$. We shall use the method of acyclic models to construct the chain homotopy in degree n for the universal example of the diagonal singular simplex $\mathrm{Diag}_n : \Delta_n \to \Delta_n \times \Delta_n$ and extend it to all singular simplices by naturality.

To construct the chain homotopy on the universal example in degree n, we need to find some $\varphi_n \in [S_*(\Delta_n) \otimes S_*(\Delta_n)]_{n+1}$ such that

$$d\varphi_n \;=\; \psi \circ T_\# \left(\mathrm{Diag}_n\right) \;-\; \tau \circ \psi \left(\mathrm{Diag}_n\right) \;-\; D \circ d \left(\mathrm{Diag}_n\right) \ .$$

Since $S_*(\Delta_n) \otimes S_*(\Delta_n)$ is acyclic, we can find a suitable element φ_n if and only if the right hand side is a cycle. As in previous examples, this can be shown using the facts that ψ, T and τ are chain maps, the chain complex identity $d \circ d = 0$, and the fact that D is a chain homotopy through degree $n-1$; details are left to the reader.■

We are now ready to state the grade-commutativity properties of the cross product and their implications for grade-commutativity of the cup product.

THEOREM 4. *Let X and Y be topological spaces, let $u \in H^p(X)$ and $v \in H^q(Y)$ be cohomology classes, and let $T : X \times Y \to Y \times X$ denote the transposition homeomorphism. Then the cohomology cross product satisfies $u \times v = (-1)^{pq} T^*(v \times u)$.*

COROLLARY 5. *If X is a space with $u \in H^p(X)$ and $v \in H^q(X)$, then $u \cup v = (-1)^{pq} v \cup u$.*

Proof of Corollary 5. If $\Delta_X : X \to X \times X$ is the diagonal, then $u \cup v = \Delta_X^*(u \times v)$, and if $T : X \times X \to X \times X$ transposes coordinates, then $T \circ \Delta_X = \Delta_X$. Therefore by Theorem 5 we have

$$u \cup v \;=\; \Delta_X^*(u \times v) \;=\; (T \circ \Delta_X)^*(u \times v) \;=\; \Delta_X^* \circ T^*(u \times v) \;=\; \Delta_X^*\left((-1)^{pq} v \times u\right) \;=\; (-1)^{pq} v \cup u$$

which is what we wanted to prove.■

Proof of Theorem 4. Choose cocycles f and g representing u and v respectively, and let $\psi_{X,Y}$ and $\psi_{Y,X}$ be the Alexander-Whitney maps for $X \times Y$ and $Y \times X$. Then we have the following diagram in which the right hand square commutes and the left hand square commutes up to chain homotopy:

$$\begin{array}{ccccc}
S_{p+q}(X \times Y) & \xrightarrow{\mathrm{proj}(p,q)\psi} & S_p(X) \otimes S_q(Y) & \xrightarrow{f \otimes g} & \mathbb{D} \otimes \mathbb{D} \cong \mathbb{D} \\
\downarrow T_{\#} & & \downarrow \tau & & \downarrow = \\
S_{p+q}(Y \times X) & \xrightarrow{\mathrm{proj}(p,q)\psi} & S_p(X) \otimes S_q(Y) & \xrightarrow{(-1)^{pq} g \otimes f} & \mathbb{D} \otimes \mathbb{D} \cong \mathbb{D}
\end{array}$$

The map $\mathrm{proj}(p,q)$ is projection onto the direct summand $S_p(X) \otimes S_q(Y)$ in $[S_*(X) \otimes S_*(Y)]_{p+q}$, while the top row is a cochain level representative for $u \times v$ and the bottom row is a cochain level representative for $(-1)^{pq} u \times v$.

Let E be the chain homotopy relating $\psi_{Y,X} \circ T_{\#}$ and $\tau \circ \psi_{X,Y}$. Then we have

$$f \times g \;=\; (f \otimes g) \circ \psi_{X,Y} \;=\; (-1)^{pq} (g \otimes f) \circ \tau \circ \psi_{X,Y} \;=\; (-1)^{pq} (g \otimes f) \circ (\psi_{Y,X} \circ T_{\#} + dE + Ed) \;=\;$$

$$T^{\#}\left((-1)^{pq} (g \otimes f) \circ \psi_{Y,X}\right) \;+\; (-1)^{pq} \delta(g \times f) \circ E \;+\; \delta U$$

where δ is the coboundary map and U is some cochain whose precise value is unimportant because it disappears when we take cohomology classes. The term $\delta(g \times f)$ vanishes because f and g are cocycles, and therefore the displayed identities show that the cohomology classes represented by $f \times g$ and $(-1)^{pq}(g \times f)$ — namely, $u \times v$ and $(-1)^{pq} T^*(v \times u)$ — must be equal.■

Some examples

The results of this and the previous section yield complete information on the cup product structure for a product of spheres with coefficients in a field. This can be done inductively using the theorem stated below. Before stating this result, we need the following construction:

Definition. Let A_* and B_* be graded algebras over $\mathbb{D}$ with multiplication maps μ_A and μ_B respectively. Then the tensor product $A_* \otimes B_*$ has a multiplication given by

$$(A_* \otimes B_*) \otimes (A_* \otimes B_*) \to A_* \otimes A_* \otimes B_* \otimes B_* \to A_* \otimes B_*$$

where the first map is the middle four interchange

$$x_p \otimes y_q \otimes z_r \otimes w_s \quad \longrightarrow \quad (-1)^{qr}\, x_p \otimes z_r \otimes y_q \otimes w_s$$

and the second map is $\mu_A \otimes \mu_B$.

THEOREM 6. *Let n be a positive integer, let $\mathbb{F}$ be a field, and let X be a space such that $H^k(X;\mathbb{F})$ is finite dimensional for all k. Then the cohomology algebra $H^*(S^n \times X;\mathbb{F})$ is isomorphic to the tensor product algebra $H^*(S^n) \otimes_{\mathbb{F}} H^*(X)$.*

This result is an immediate consequence of the cohomological Künneth Theorem; details are again left to the reader.■

The preceding theorem and induction yield the computation for the cohomology of a product

$$\prod_{k=1}^{r} S^{n(k)}$$

where each $n(k)$ is positive.

COROLLARY 7. *In the setting above we have*

$$H^*\left(\prod_{k=1}^{r} S^{n(k)};\mathbb{F}\right) \quad \cong \quad \bigotimes_{k=1}^{r} H^*\left(S^{n(k)};\mathbb{F}\right) \;.■$$

The following results are immediate consequences of this corollary:

COROLLARY 8. *In the setting above, assume that for each k the cohomology class u_k is the image of a generator for $H^{n(k)}(S^{n(k)};\mathbb{F})$ under the map induced by the coordinate projection p_k onto the k^{th} factor. Then*

$$\prod_k u_k \quad \neq \quad 0\;.$$

This is merely an iteration of the fact that the cross product of two nontrivial cohomology classes is always nontrivial.■

COROLLARY 9. *In the setting above, assume that the dimensions $n(k)$ are all even and equal to some fixed integer n (hence the cup product is commutative in the usual sense), and assume further that for each k the cohomology class u_k is the image of a generator for $H^(S^n;\mathbb{F})$ under the map induced by the coordinate projection p_k onto the k^{th} factor. Then*

$$\left(\sum_k u_k\right)^r \quad \neq \quad 0\;.$$

This reduces to a purely algebraic computation, which shows that the class in question is equal to $n! \cdot \prod_k\ u_k$ (the details are again left as an exercise$^{(\star)}$).■

IV.5 : Two applications

(Hatcher, §§ 3.2, 4.2)

Although we have only obtained relatively weak versions of the basic results on products in singular homology and cohomology theory, they suffice to yield two fairly significant results. One is a restriction on the maps in homology associated to a homotopy self-equivalence from $S^{2m} \times S^{2m}$ to itself, and the other is a proof that for all $m > 1$ there is a continuous mapping from S^{4m-1} to S^{2m} which is not homotopic to a constant. The existence of such maps reflects several of the fundamental difficulties one encounters when trying to study homotopy theory.

Coefficient homomorphisms in singular homology and cohomology

We would like to have some way of extracting information about homology and cohomology groups with integer coefficients from computations of homology and cohomology with coefficients in various fields (not surprisingly, the usual examples are the rationals $\mathbb{Q}$ and the prime fields $\mathbb{Z}_p$ where p is prime). There are two or three main ideas.

(1) If $\varphi : \mathbb{D} \to \mathbb{E}$ is a homomorphism of commutative rings with unit, then there are associated natural *coefficient homomorphisms* $\varphi_{\#}$ and $\varphi^{\#}$ of singular chain and cochain groups. These are compatible with the cup and cap product structures, and they induce corresponding natural transformations in homology and cohomology which commute with the connecting homomorphisms in the long exact sequences of pairs and Mayer-Vietoris exact sequences.

(2) Let $A \to A_{(0)}$ be the rationalization functor on abelian groups which is defined in Section VII.5 of `algtop-notes.pdf`. Then there is a natural isomorphism of ∂-functors from $H_*(X, A; \mathbb{Z})_{(0)}$ to $H_*(X, A; \mathbb{Q})$ which commutes with the connecting homomorphisms in the long exact sequences of pairs and Mayer-Vietoris exact sequences.

(3) The Universal Coefficient Theorems in Hatcher provide the "right" way of extracting information about homology and cohomology groups with integer coefficients from computations involving $\mathbb{Z}_p$ coefficients. — We are avoiding this to minimize the amount of algebraic machinery developed in the course.

The first of these is easy to show; given a pair of spaces (X, A), the natural map $S_*(X, A) \otimes \mathbb{D} \to S_*(X, A) \otimes \mathbb{E}$ is just the tensor product of the identity on $S_*(X, A)$ with φ, and the map on cochains takes $f : S_*(X, A) \otimes \mathbb{D}$ to $f \circ \varphi$. All of the assertions about these maps then follow by purely formal considerations. The second principle follows immediately from Corollary VII.5.3 in `algtop-notes.pdf`.■

Similar considerations hold for simplicial chain and cochain groups, and this is true for groups defined with respect to orderings of vertices and groups defined without such orderings. Furthermore, the coefficient homomorphisms commute with the maps defined by passage from ordered to unordered simplicial chains, and from unordered simplicial chains to singular chains.■

Cellular homology and cohomology with coefficients. Let $(X, \mathcal{E})$ be a finite cell complex, and let X_k denote the k-skeleton of this complex. Then one can define cellular chain groups with coefficients in an arbitrary commutative ring with unit $\mathbb{D}$, and the proof given in the integer case extends directly to show that the groups $H_*(X; \mathbb{D})$ are isomorphic to the homology groups of the complex $C_*(X, \mathcal{E}; \mathbb{D})$. One can also define cellular cochain complexes $C^*(X, \mathcal{E}; \mathbb{D})$ such that

$C^k(X, \mathcal{E}; \mathbb{D}) = H^k(X_k, X_{k-1}; \mathbb{D})$ — which will be a free abelian group whose rank is the number of k-cells — and a dualization of the earlier arguments shows that the cohomology of X with coefficients in $\mathbb{D}$ is isomorphic to the cohomology of the cellular cochain complex (the details are left to the reader).

It is not difficult to guess how cellular chain and cochain complexes behave under the coefficient homomorphism associated to a ring homomorphism $\varphi : \mathbb{D} \to \mathbb{E}$:

PROPOSITION 0. *In the setting above, let $F(\mathbb{D})$ denote the chain or cochain group $C_k(X, \mathcal{E}; \mathbb{D})$ or $C^k(X, \mathcal{E}; \mathbb{D})$, and let $\varphi_* : F(\mathbb{D}) \to F(\mathbb{E})$ be the coefficient map induced by φ. Use the standard free generators for the chain or cochain group (corresponding to the k-cells in X) to identify $F(\mathbb{D})$ and $F(\mathbb{E})$ with $F(\mathbb{Z}) \otimes \mathbb{D}$ and $F(\mathbb{Z}) \otimes \mathbb{E}$ respectively. Then the coefficient homomorphism φ_* corresponds to $\mathbf{id}[F(\mathbb{Z})] \otimes \varphi$.*

Sketch of proof. By naturality considerations it suffices to prove the analogous result for the homology of (D^k, S^{k-1}). This case can be treated explicitly using the ordered simplicial chain complex for the pair $(\Delta_k, \partial\delta_k)$.■

Cell decompositions for products of spheres

Let n be a positive integer, and let $\mathbb{D}$ be a commutative ring with unit.

If we take the simplest cell decomposition for S^n with a 0-cell and an n-cell, then the product construction yields a cell decomposition of $S^n \times S^n$ with one 0-cell, two n-cells and one $2n$-cell. If $n \geq 2$ then there are no possible nonzero differentials in the cellular chain complex for computing $H_*(S^n \times S^n; \mathbb{D})$ and hence one can read off the homology immediately. If $\sigma \in H_n(S^n; \mathbb{D}) \cong \mathbb{D}$ is a generator and i_1, i_2 are the usual slice inclusions, then the classes $i_{1*}\sigma$ and $i_{2*}\sigma$ form a free basis for $H_n(S^n \times S^n; \mathbb{D})$. The top cell of this complex is attached to the n-skeleton, which is a wedge of two copies of S^n by a continuous map

$$P : S^{2n-1} \longrightarrow S^n \vee S^n$$

that we shall call the *universal Whitehead product.*

Let n be as in the preceding paragraph, and let $PT^n \subset T^n$ denote the $(n-1)$-skeleton with respect to the standard cell decomposition of T^n described earlier. Then the quotient space T^n/PT^n is homeomorphic to S^n; let $\kappa : T^n \to S^n$ denote the associated collapsing map. It follows that κ_* and κ^* induce isomorphisms in n-dimensional homology and cohomology (say with field coefficients in the second case). Furthermore, it follows that $\kappa \times \kappa : T^n \times T^n \to S^n \times S^n$ induces a monomorphism in cohomology; verifying this is a fairly straightforward exercise using the corresponding property of κ^*, the known structure of $H^*(S^n \times S^n)$, and the known structure of $H^*(T^n \times F^n)$.

The preceding discussion reduces the computation of the cohomology cup product for $S^n \times S^n$ to questions about the corresponding structure for $T^{2n} = T^n \times T^n$. Here is a formal statement of the conclusions:

PROPOSITION 1. *Let $\Omega \in H^n(S^n)$ be such that the Kronecker index $\langle \Omega, \sigma \rangle = 1$, and let π_1, π_2 denote the projections of $S^n \times S^n$ onto the factors. Then the cohomology classes $\pi_j^*\Omega$ are dual to the homology classes $i_{j*}\sigma$ with respect to the Kronecker index pairing, and these classes satisfy the following conditions:*

(*i*) *Their cup squares are zero.*

(*ii*) *The class* $\pi_1^*\Omega \cup \pi_2^*\Omega$ *generates* $H^{2n}(S^n \times S^n)$.

Proof. In the cellular decomposition for $S^n \times S^n$ described above, there are no cells in adjacent dimensions, and hence the cellular chain and cochain complexes have trivial differentials. Thus the cellular decomposition and the discussion of cellular cohomology with coefficients imply that $H^k(S^n \times S^n; \mathbb{D})$ is isomorphic to $\mathbb{D} \oplus \mathbb{D}$ if $k = n$, $\mathbb{D}$ if $k = 0$ or $2n$, and zero otherwise. Furthermore, by construction $H^n(S^n \times S^n; \mathbb{D})$ is freely generated by the classes $\pi_1^*\Omega_{\mathbb{D}}$ and $\pi_2^*\Omega_{\mathbb{D}}$, where $\Omega_{\mathbb{D}}$ is the image of Ω under the coefficient homomorphism $\varphi : \mathbb{Z} \to \mathbb{D}$ sending 1 to the identity in $\mathbb{D}$.

The first conclusion holds because Ω^2 in the cohomology of S^n and the maps π_t^* are multiplicative. To prove the second statment, let p be a prime and take $\mathbb{D} = \mathbb{Z}_p$. Then the Künneth Theorem for cohomology implies that $\pi_1^*\Omega_{\mathbb{D}} \cup \pi_2^*\Omega_{\mathbb{D}}$ generates $H^{2n}(S^n \times S^n; \mathbb{Z}_p) \cong \mathbb{Z}_p$. Therefore by Proposition 0 the image of $\pi_1^*\Omega \cup \pi_2^*\Omega$ in $\mathbb{Z}_p$ is a generator for all primes p, and it follows that $\pi_1^*\Omega \cup \pi_2^*\Omega$ must be a generator for $H^{2n}(S^n \times S^n) \cong \mathbb{Z}$ (otherwise, its image in some $\mathbb{Z}_p$ would be trivial).

Finally, we also note that if n is even then grade-commutativity of cup products implies that $\pi_1^*\Omega \cup \pi_2^*\Omega = \pi_2^*\Omega \cup \pi_1^*\Omega$.■

These computations lead directly to our first application.

THEOREM 2. *Suppose that* $m \geq 1$ *and* f *is a homotopy self-equivalence of* $S^{2m} \times S^{2m}$. *Let* σ_1 *and* σ_2 *denote the free basis for* $H_{2m}(S^{2m} \times S^{2m}; \mathbb{Z})$ *described earlier. Then either the associated map in homology* f_* *sends the* σ_j *to* $\varepsilon_j\sigma_j$, *where* $\varepsilon_j = \pm 1$, *or else* f_* *sends* σ_1 *to* $\varepsilon_1\sigma_2$ *and sends* σ_2 *to* $\varepsilon_1\sigma_1$ *where again* $\varepsilon_j = \pm 1$.

All of the possibilities in the theorem can be realized. For the first alternatives this can be done by considering various product of the form 1, $1 \times \rho$, $\rho \times 1$ and $\rho \times \rho$, where ρ is the reflection involution on a sphere, and the second alternatives can be realized by composing the first alternatives with the transposition map τ on $S^{2m} \times S^{2m}$.

Suppose now that n is an arbitrary positive integer. Since $H_n(S^n \times S^n; \mathbb{Z}) \cong \mathbb{Z}^2$, the only general algebraic restriction one can get on a map f_* induced by a homotopy self-equivalence is that it must correspond to a 2×2 matrix over the integers with determinant equal to ± 1. It is fairly simple to construct examples of homotopy self equivalences of T^2 which realize every such matrix (the associated linear transformations of $\mathbb{R}^2$ pass to homeomorphisms of T^2). If n is odd, then the possible 2×2 matrices are also understood, but this is a deeper result; the conclusion is that one can realize every matrix if $n = 1, 3, 7$, while for the remaining odd values of n it is possible to realize every integral 2×2 matrix with determinant ± 1 whose reduction mod 2 is a permutation matrix. For the exceptional odd values of n, one can show this using standard "multiplications" on S^n (given by restricting complex, quaternionic, and Cayley number multiplication to the unit sphere in $\mathbb{R}^{n+1}$ where $n + 1 = 2, 4, 8$). For the remaining odd values of n, this fact is due to J. F. Adams and was proved in the nineteen fifties. Here are (very terse) references for the latter:

`http://mathworld.wolfram.com/H-Space.html`

`http://mathworld.wolfram.com/HopfInvariantOneTheorem.html`

Proof of Theorem 2. As noted in the preceding paragraph, if σ_1 and σ_2 are the given standard free basis for $H_{2m}(S^{2m} \times S^{2m}; \mathbb{Z}) \cong \mathbb{Z}^2$, then there are integers a, b, c, d such that $ad - bc = \pm 1$ and $f_*(\sigma_1) = a\sigma_1 + b\sigma_2$, $f_*(\sigma_2) = c\sigma_1 + d\sigma_2$. By the naturality of homology with respect to coefficient homomorphisms, it follows that one has a similar description of f_* with rational coefficients. If we take the dual basis ξ_1, ξ_2 of $H^{2m}(S^{2m} \times S^{2m}; \mathbb{Q})$, then it follows that $f^*\xi_1 = a\xi_1 + c\xi_2$ and

$f^*\xi_2 = b\xi_1 + d\xi_2$. Since f preserves cup products and $\xi_j^2 = 0$, the same is true for $f^*(\xi_j)$. But Proposition 1 implies that

$$f^*(\xi_1)^2 \;=\; 2ac\,\xi_1 \cup \xi_2 \;, \qquad f^*(\xi_2)^2 \;=\; 2bd\,\xi_1 \cup \xi_2$$

and since $\xi_1 \cup \xi_2$ is nonzero it follows that $ac = bd = 0$, so that either $a = 0$ or $c = 0$ and also either $b = 0$ and $d = 0$. The cases $a = b = 0$ and $c = d = 0$ both imply that $ad - bc = 0$, so neither can hold, and therefore the only possibilities are $a = d = 0$ or $c = b = 0$. In the first case the condition $ad - bc$ implies that $b, c \in \{\pm 1\}$, while in the second case we must have $a, d \in \{\pm 1\}$. These are precisely the options listed in the theorem.■

Homotopically nontrivial mappings of spheres

If $m < n$ then simplicial approximation implies that every continuous mapping from S^m to S^n is homotopically trivial, and if $m = n$ we know that there are infinitely many homotopy classes of maps $S^n \to S^n$ which can be distinguished homotopically by their degrees; we have not proved that two maps of the same degree are homotopic, but it would not be exceedingly difficult for us to do so at this point (for example, see the argument in Maunder, *Algebraic Topology*, pages 288–291; the statement of this result in Hatcher is Corollary 4.25 on page 361). The important point is that if $m \leq n$, then homotopy classes of maps from S^m to S^n can be distinguished using homology theory. Given that every map from S^m to S^1 is nullhomotopic if $m > 1$, it was natural to hope that all maps $S^m \to S^n$ would be homotopic to constant maps. However, counterexamples began to surface during the nineteen thirties, and describing the homotopy classes of mappings from S^{n+k} to S^n where $k > 0$ turns out to be an exceedingly difficult problem, although it is known that the answer for any specific choice of n and k is finitely computable (although the basic algorithm seems unlikely to be implemented in the foreseeable future). We shall limit ourselves to a single class of important examples:

THEOREM 3. *Suppose that m is a positive integer. Then there is a continuous mapping $f : S^{4m-1} \to S^{2m}$ which is not homotopic to a constant.*

In fact, refinements of our methods show that there are infinitely many distinct homotopy classes of such maps. There is actually a very striking converse to this result discovered by J.-P. Serre in the nineteen fifties:

> *For all $m, n > 0$, there are only finitely many homotopy classes of continuous mappings from S^n to S^m* **unless** *$m = n$ or m is even and $n = 2m - 1$.*

One reference for this result is Section 9.7 of Spanier. The basic reference for the finite computability statement is the following paper;

> **E.** (= *Edgar*) **H. Brown.** *Finite computability of Postnikov complexes.* Annals of Mathematics **65** (1957), 1–20.

Proof. Throughout this discussion the coefficient field will be the rational numbers $\mathbb{Q}$.

The examples will be composites of the form $\nabla \circ P$, where $P : S^{4m-1} \to S^{2m} \vee S^{2m}$ is the universal Whitehead product described earlier and $\nabla : S^{2m} \vee S^{2m} \to S^{2m}$ folds the two wedge summands together (so its restriction to each summand is the identity). This class is generally known as the *Whitehead product* of the identity map on S^{2m} with itself and denoted by $[\iota_{2m}, \iota_{2m}]$ (compare Hatcher, Example 4.52, page 381). The argument wll require the following relatively elementary observation:

LEMMA 4. *Suppose that $f : S^{p-1} \to A$ is a continuous map into a compact metric space and X is the space obtained by attaching a p-cell to A along f. If f is homotopic to a constant map, then the inclusion of A in X is a retract.*

Proof of Lemma 4. If f is homotopic to a constant, then f extends to a mapping $g : D^p \to A$. Write $X = A \cup E$, where E is the p-cell. Then the retraction from X to A is defined by taking the identity on A and using g to define the mapping on E. By construction, it follows that these definitions fit together to yield a well-defined continuous retraction from X to A.∎

Returning to the proof of Theorem 3, let $K(f)$ be the space obtained by adjoining a $4m$-cell to S^{2m} along the mapping $\nabla \circ P$. We then have the following commutative diagram, in which the two horizontal arrows on the left are attaching maps, the middle horizontal arrows are inclusions, and the horizontal arrows on the right are maps which collapse the codomains of the attaching maps to points.

$$\begin{array}{ccccccc} S^{4m-1} & \xrightarrow{P} & S^{2m} \vee S^{2m} & \longrightarrow & S^{2m} \times S^{2m} & \longrightarrow & S^{4m} \\ \downarrow = & & \downarrow \nabla & & \downarrow h & & \downarrow = \\ S^{4m-1} & \xrightarrow{\nabla P} & S^{2m} & \longrightarrow & K(f) & \longrightarrow & S^{4m} \end{array}$$

This diagram yields the following commutative diagrams in cohomology for each $q > 0$; the rows of these diagrams are short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^q(S^{4m}) & \longrightarrow & H^q(K(f)) & \longrightarrow & H^q(S^{2m}) & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow h^* & & \downarrow \nabla^* & & \\ 0 & \longrightarrow & H^q(S^{4m}) & \longrightarrow & H^q(S^{2m} \times S^{2m}) & \longrightarrow & H^q(S^{2m} \vee S^{2m}) & \longrightarrow & 0 \end{array}$$

It follows that $H^*(K(f))$ is isomorphic to $\mathbb{Q}$ in dimensions $0, 2m, 4m$ and is trivial otherwise. Let θ denote a generator for $H^{2m}\big(K(f)\big)$. It follows that $h^*(\theta)$ is a nonzero multiple of $\xi_1 + \xi_2$, and we might as well choose θ so that it maps to this class in $H^{2m}(S^{2m} \times S^{2m})$. Furthermore, we have

$$h^*(\theta)^2 \quad = \quad 2\,\xi_1 \cup \xi_2 \quad \neq 0$$

so that θ^2 must also be nonzero in $H^{4m}\big(K(f)\big)$.

We claim that the statement in the preceding sentence implies that f cannot be nullhomotopic. If it were, then there would be a retraction $\rho : K(f) \to S^{2m}$, and θ would have to be in the image of ρ^*. But if $\theta = \rho^*\theta_0$ for some $\theta_0 \in H^*(S^{2m})$, then $\theta_0^2 = 0$ and hence $\theta^2 = 0$, contradicting the conclusions in the preceding paragraph. Hence the only possibility consistent with the latter is that f is not nullhomotopic.∎

IV.6 : Open disk coverings of manifolds

(Hatcher, § 3.2)

Every compact topological n-manifold is a union of fintely many open subsets U_i such each U_i is homeomorphic to $\mathbb{R}^n$. Since each such open subset is noncompact, it is clear that one needs at least two such open subsets, and of course S^n is an example where the minimum number is exactly two. More generally, one can ask the following question:

Suppose that X is a compact Hausdorff space which has at least one open covering consisting entirely of contractible sets. What is the MINIMUM *number of such sets that are needed to form an open covering of X?*

If X is a topological n-manifold, then the following basic result gives an upper estimate:

THEOREM 1. *If M is a (second countable) arcwise connected topological n-manifold, then M has an open covering by $n+1$ sets which are homeomorphic to open subsets of $\mathbb{R}^n$.*■

Here is the standard reference for a proof:

E. Luft. *Covering manifolds with open cells.* Illinois Journal of Mathematics **13** (1969), 321–326.

In this section we shall use cup products to prove that in general the best possible upper bound is $n+1$.

Lusternik-Schnirelmann category and cup products

There are numerous variants of the contractible open covering question, and we shall be particularly interested in a version where "contractible open sets" is replaced by "open subsets for which the inclusion maps into X are nullhomotopic." In particular, the following homotopy-theoretic concept is closely related to these questions:

Definition. Let X be a second countable, locally compact, Hausdorff space. Then X is said to have *Lusternik-Schnirelmann* or **LS** *category* $\leq m$ if X is a union of m subsets U_i such that the inclusions $U_i \subset M$ are nullhomotopic.

Note. Frequently one finds slightly different spellings of the names "Lusternik" and "Schnirelmann" based upon different conventions for transliterating the Cyrillic spellings Люстерник and Шнирельман into their Latin counterparts.

Definition. We shall say that X has Lusternik-Schnirelmann or **LS** category equal to k if it has **LS** category $\leq k$ but does not have **LS** category $\leq k-1$. Similarly, we shall say that X has **LS** category $\geq k$ if X does not have **LS** category $\leq k-1$.

If X is a compact topological n-manifold which has a covering by k open subsets, each homeomorphic to $\mathbb{R}^n$, then it follows immediately that X has **LS** category $\leq k$, and Theorem 1 implies that the **LS** category is always $\leq n+1$. The main result of this section gives an example where equality holds.

THEOREM 2. *The n-torus T^n has* **LS** *category equal to $n+1$.*

The proof that T^n has **LS** category $\geq n+1$ will be a consequence of the following general observation.

THEOREM 3. *Suppose that X is an arcwise connected, second countable, locally compact, Hausdorff space with* **LS** *category $\leq m$, and let $u_1 \in H^{d(1)}(X,\mathbb{F})$, $\cdots$ $,u_m \in H^{d(m)}(X;\mathbb{F})$ with $d(i) > 0$ for all i. Then $u_1 \ \cdots \ u_m = 0$.*

If the conclusion of the theorem holds for an arcwise connected space X, we shall say that X *has* cuplength $\leq m$ because every product of m positive-dimensional cohomology classes in X is equal to zero.

Proof of Theorem 3. Let $W_1, \ \cdot \ , W_m$ be a covering of X such that each inclusion $W_i \to X$ is nullhomotopic. Since each cohomology restriction map $H^{m(i)}(X;\mathbb{F}) \to H^{m(i)}(W_i;\mathbb{F})$ is trivial, the classes u_i lift to classes v_i in the relative cohomology groups $H^{m(i)}(X, W_i; \mathbb{F})$. It follows that $u_1 \cdots u_m$ is the image of $v_1 \cdots v_m$ is the group

$$H^*(X, \cup_i U_i; \mathbb{F}) \quad = \quad H^*(X, X; \mathbb{F}) \quad = \quad 0$$

and hence this product equals zero.■

Proof of Theorem 2. Since there are n classes in $H^1(T^n; \mathbb{F})$ whose cup product is nonzero, Theorem 3 implies that T^n has **LS** category $\geq n+1$, and hence every open covering of T^n by sets homeomorphic to $\mathbb{R}^n$ consists of at least $n+1$ such regions.

One can construct an explicit open covering of T^n with $n+1$ open sets as follows: Let $p : \mathbb{R}^n \to T^n$ be the usual universal covering projection sending $(t_1, \cdots, t_n)$ to $(\exp 2\pi\, i\, t_1, \cdots, \exp 2\pi\, i\, t_n)$, and let $a_0, \cdots, a_n$ be distinct points in the half-open interval $[0, 1)$, so that the points $z_k = \exp 2\pi\, i\, a_k \in S^1$ are distinct. Now let $W_k \subset \mathbb{R}^n$ be the set of all points such that $a_k < t_k < a_k + 1$ for all k, and take $V_k \subset T^n$ to be the image of W_k under p. By construction each set V_k is contractible. A point of T^n will lie in $T^n - V_k$ if and only if at least one of its coordinates is equal to z_k. The intersection of the sets $T^n - V_k$ will consist of all points $(b_1, \cdots, b_n)$ such that for each k, there is some j for which $b_j = z_k$. Since there are $n+1$ values of z_k and only n coordinates b_j, this is impossible. Therefore $\cap_k\ (T^n - V_k) \ = \ \emptyset$, so that $T^n \ = \ \cup_k\ V_k$.

References for further information

The *Wikipedia* article

`http://en.wikipedia.org/wiki/Lusternik%E2%80%Schnirelmann_category`

is a good starting point for learning more about the concept of Lusternik-Schnirelmann category, and it gives several good references for further information on the topic. The book by Cornea, Lupton, Oprea and Tanré (cited in that article) contains a very thorough treatment of this subject.

IV.7 : Real and complex projective spaces

(Hatcher, Ch. 0 and §§ 1.2–1.3, 2.2, 2.C, 3.2)

See the files `projspaces*.pdf` where $* = 1$ or 2.

V. Cohomology and Differential Forms

Courses in multivariable calculus generally end with proofs of fundamental results in vector analysis such as Green's Theorem in the plane, path independence criteria for line integrals, Stokes' Theorem for oriented surfaces with boundaries, and the Divergence Theorem in 3-space. Differential forms provide the standard framework for stating and proving the corresponding results in higher dimensions. We have already seen that algebraic topology also provides a setting in which various global versions of these results can be formulated. These included comprehensive generalizations of Green's Theorem and the Divergence Theorem to regions which have nice decompositions. The purpose of this unit is to describe far-reaching extensions of such relationships to arbitrary finite dimensions. In particular, we shall see that the answer to the question

> *Are there k-dimensional differential forms on an open subset U of $\mathbb{R}^n$ which are closed ($d\omega = 0$) but not exact ($\omega = d\theta$ for some θ)?*

depends only whether or not the singular homology group $H_k(U;\mathbb{R})$ is trivial (in which case the answer is no) or nontrivial (in which case the answer is yes). This even yields new information in the setting of classical vector analysis; specifically, if U is an open subset in $\mathbb{R}^3$, then every smooth vector field $\mathbf{F}$ whose curl satisfies $\nabla \times \mathbf{F} = \mathbf{0}$ is a gradient vector field if and only if $H_1(U;\mathbb{R}) = 0$. This is one of many corollaries of a fundamental result known as **de Rham's Theorem**.

Section 0 is a summary of the main things we need to know about differential k-forms on an open subset of $\mathbb{R}^n$. Roughly speaking, these are formal integrands of line integrals, surface integrals, multiple integrals, and their generalizations to integrals over suitably defined k-dimensional analogs of surfaces in $\mathbb{R}^n$. In Section 1 we make the latter notion precise by defining a variant of singular homology in which the singular simplices are smooth mappings, and in Section 2 we state an analog of Stokes' Theorem for the integral of a k-form over a k-dimensional smooth singular chain. Differentiation of differential forms induces maps d^k from k-forms to $(k+1)$-forms which satisfy $d^{k+1} \circ d^k = 0$, and thus the differential forms on an open subset in $\mathbb{R}^n$ form a cochain complex often called the *de Rham complex* of an open set. The cohomology groups of this cochain complex are called the *de Rham cohomology groups* of the open set, and Section 3 shows that these groups have several formal properties which resemble those of singular cohomology groups with real coefficients. The main result of Section 4 is de Rham's theorem, which states that the two types of cohomology groups are isomorphic. Finally, in Section 5 we shall prove that under this isomorphism the cup product in singular cohomology corresponds to a construction on differential forms known as the wedge product.

Throughout the rest of this section we shall refer to the following textbook for the details of various constructions and proofs:

> **L. Conlon.** Differentiable Manifolds. (Second Edition), *Birkhäuser-Boston, Boston MA*, 2001. ISBN: 0–8176–4134–3.

There will also be references to Lee's book on smooth manifolds; however, in many instances the discussion in Lee is at a more abstract and general level than these notes (in particular, it gets into some complicated issues that we are trying to avoid).

V.0 : Review of differential forms

(Conlon, §§ 6.2, 6.4, 7.1–7.2, 8.1; Lee, Chs. 6, 11-12)

We have already noted that differential forms provide a convenient and powerful setting for generalizing classical vector analysis to higher dimensions, but they also have numerous other uses in both mathematics and physics. Setting up the theory requires some time and effort, but differential forms can be used very effectively to unify and simplify some fundamentally important concepts and results. They have become the standard framework for analyzing an extremely wide range of topics and problems. For the most part, we shall restrict attention to differential forms on open subsets of $\mathbb{R}^n$ where n is allowed to be a more or less arbitrary positive integer.

This is only a summary of the main points of the theory. Additional details can be found on pages 245–288 of Rudin (*Principles of Mathematical Analysis*, Third Edition).

Covariant tensors and differential forms

Let U be an open subset of $\mathbb{R}^n$, and let p be a nonnegative integer. A **covariant tensor field of rank** p is defined to be an expression of the form

$$\sum_{i_1, i_2,\ (etc.)} g_{i_1\, i_2\, \cdots\, i_p}\, dx^{i_1} \otimes\ \cdots\ \otimes dx^{i_p}$$

where

(1) each $g_{i_1\, i_2\, \cdots\, i_p}$ is a smooth real valued function on U,

(2) each i_j ranges from 1 to n,

(3) two expressions are equal if and only if the functional coefficients of each $dx^{i_1} \otimes \cdots \otimes dx^{i_p}$ are equal.

We shall call denote this object by $\mathbf{Cov}_p(U)$. It will be understood that $\mathbf{Cov}_0(U) = \mathcal{C}^\infty(U)$; note also that there is a natural identification of $\mathbf{Cov}^1(U)$ with the space of differential 1-forms we considered in Section V.3 of the lecture notes.

The space of **exterior** or **differential** p**-forms** on U is defined to be the quotient of $\mathbf{Cov}_p(U)$ obtained by the identification

$$dx^{i_1} \otimes\ \cdots\ \otimes dx^{i_p} \quad \approx \quad -\, dx^{j_1} \otimes\ \cdots\ \otimes dx^{j_p}$$

if $[\, j_1\, j_2\ \cdots\ j_p\,]$ is obtained from $[\, i_1\, i_2\ \cdots\ i_p\,]$ by switching exactly two of the terms, say i_s and i_t where $s \neq t$. If $i_s = i_t$ for some $s \neq t$ then this is understood to imply that $dx^{i_1} \otimes\ \cdots\ \otimes dx^{i_p}$ is equal to its own negative, and since we are working with real vector spaces this means that the expression in question is identified with zero. The set of all differential p-forms on an open subset $U \subset \mathbb{R}^n$ is denoted by $\wedge^p(U)$, and the images of the basic objects in if $dx^{i_1} \otimes\ \cdots\ \otimes dx^{i_p}$ is one of the basic objects in $\mathbf{Cov}_p(U)$ as above, then its image in $\wedge^p(U)$ is denoted by

$$dx^{i_1} \wedge\ \cdots\ \wedge dx^{i_p}\ .$$

By convention we also set $\wedge^0(U)$ equal to $\mathcal{C}^\infty(U)$.

PROPOSITION 1. *If $p > n$ then $\wedge^p(U) = 0$, and if $0 < p \leq n$ then every element of $\wedge^p(U)$ can be written uniquely as a linear combination of the basic forms*

$$dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

with coefficients in $\mathcal{C}^\infty(U)$, where the indexing sequences $\{ i_j \}$ satisfy $i_1 < \cdots < i_p$.

This is an immediate consequence of the construction.■

If $p = 1$ then the definition of $\wedge^1(U)$ is equivalent to the previous one involving sections of the cotangent bundle.

Integrals defined by differential forms The motivation for the definition comes from the use of differential 1-forms as the integrands of line integrals. In particular, we would like 2-forms to represent the integrands of surface integrals and n-forms to represent the integrands of ordinary (Riemann or Lebesgue) integrals over appropriate subsets of U. Note in particular that if U is open in $\mathbb{R}^n$, then every element of $\wedge^n(U)$ is uniquely expressible as

$$h(x) \cdot dx^1 \wedge \cdots \wedge dx^n$$

for some $h \in \mathcal{C}^\infty(U)$.

So how do we form integrals such that the integrand is a p-form and the construction reduces to the usual ones for line and surface integrals if $p = 1$ or 2? The key is to notice that such integrals are first defined using parametric equations for a curve or surface defined for all values of the variable(s) in some open subset of $\mathbb{R}$ or $\mathbb{R}^2$.

Following Rudin, we do so by defining a *smooth singular p-surface piece* in U to be a continuous map $\sigma : \Delta \to U$ such that Δ is compact in $\mathbb{R}^p$ and σ extends to a smooth function on an open neighborhood of Δ in $\mathbb{R}^p$. In multivariable calculus one generally assumes also that the extension of σ to an open set is a *smooth immersion*, or at least this is true if one subdivides the domain of definition into suitable pieces and permits bad behavior at boundary points of such pieces (normally the boundary has measure zero and hence doesn't matter for integration purposes), but we shall not make any such assumptions on the rank of $D\sigma$ in these notes.

For each object σ as in the previous paragraph and each tensor $\Lambda \in \mathbf{Cov}_p(U)$ we can define an integral by the following formula:

$$\int_\sigma \Lambda \quad = \quad \int_\sigma \sum_{i_1, i_2, \ etc.} g_{i_1\, i_2\, \cdots\, i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p} \quad =$$

$$\sum_{i_1, i_2, \ etc.} \int_\Delta g_{i_1\, i_2\, \cdots\, i_p} \circ \sigma(u) \frac{\partial(x^{i_1}, \cdots, x^{i_p})}{\partial(u^1, \cdots, u^p)}$$

As usual, expressions of the form

$$\frac{\partial(x_a, \cdots)}{\partial(u_1, \cdots)}$$

represent Jacobian determinants. We then have the following key observation which allows us to work with forms rather than tensors:

PROPOSITION 2. *In the integral above, the value only depends upon the image λ of Λ in $\wedge^p(U)$.*

Proof. It suffices to consider simple integrands consisting of only one summand. For each sequence

$$x^{i_1}, \cdots, x^{i_p}$$

we need to show that if we switch two terms x^a and x^b then the sign of the integral changes if dx^a and dx^b are both factors of the integrand. The effect of making such a change on the integrand is to switch two columns in the $p \times p$ matrix of functions whose determinant is the Jacobian

$$\frac{\partial(x^{i_1}, \cdots, x^{i_p})}{\partial(u^1, \cdots, u^p)}$$

and we know this operation changes signs; this proves the point that we need to reach the conclusion of the proposition.■

Because of the preceding result we shall assume henceforth that integrands are differential p-forms.

Operations on differential forms

There are several fundamental constructions on differential forms that are used extensively.

Exterior products. It follows immediately from the definitions that each $\wedge^p(U)$ is a real vector space and in fact is a module over $\mathcal{C}^\infty(U)$ However, there is also an important multiplicative structure that we shall now describe. We shall begin by defining a version of this structure for covariant tensors. Specifically, there are $\mathcal{C}^\infty(U)$-bilinear maps

$$\otimes : \mathbf{Cov}_p(U) \times \mathbf{Cov}_q(U) \longrightarrow \mathbf{Cov}_{p+q}(U)$$

sending a pair of monomials

$$\left(g_{i_1\, i_2\, \cdots\, i_p}\, dx^{i_1} \otimes \cdots \otimes dx^{i_p} \;,\; h_{j_1\, j_2\, \cdots\, j_q}\, dx^{j_1} \otimes \cdots \otimes dx^{j_q} \right)$$

to the monomial

$$g_{i_1\, i_2\, \cdots\, i_p} h_{j_1\, j_2\, \cdots\, j_q} \cdot dx^{i_1} \otimes \cdots \otimes dx^{i_p} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q} \ .$$

In order to show this passes to a $\mathcal{C}^\infty(U)$-bilinear map

$$\wedge_{p,q} : \wedge^p(U) \times \wedge^q(U) \longrightarrow \wedge^{p+q}(U)$$

we need to show that if $\xi \in \mathbf{Cov}_p(U)$ and $\eta \in \mathbf{Cov}_q(U)$ are monomials as above and ξ' and η' are related to ξ and η as in the definition of differential forms, then the images of $\otimes(\xi, \eta)$ and $\otimes(\xi', \eta')$ are equal. As above we are assuming

$$\xi \;=\; g_{i_1\, i_2\, \cdots\, i_p}\, dx^{i_1} \otimes \cdots \otimes dx^{i_p} \quad , \quad \eta \;=\; h_{j_1\, j_2\, \cdots\, j_q}\, dx^{j_1} \otimes \cdots \otimes dx^{j_q} \ .$$

Since two covariant monomial tensors determine the same differential form if they are related by a finite sequence of elementary moves (permuting the dx^q's or replacement by zero if there is a repeated such factor), it is enough to show that one obtains the same differential form provided ξ' and η' are related to ξ and η by a single elementary move (which affects one form but not the other).

Suppose the elementary move switches two variables; then we may write

$$\xi' \;=\; \alpha\cdot g_{k_1\,k_2\,\cdots\,k_p}\,dx^{k_1}\otimes\;\cdots\;\otimes dx^{k_p} \quad , \quad \eta' \;=\; \beta\cdot h_{\ell_1\,\ell_2\,\cdots\,\ell_q}\,dx^{\ell_1}\otimes\;\cdots\;\otimes dx^{\ell_q}$$

where $\{k_1\,k_2\,\cdots\,k_p\}$ and $\{\ell_1\,\ell_2\,\cdots\,\ell_q\}$ are obtained from $\{i_1\,i_2\,\cdots\,i_p\}$ and $\{j_1\,j_2\,\cdots\,j_q\}$ either by doing nothing or by switching two of the variables and the coefficients α and β are ± 1 depending upon whether or not variables were switched in each case. From this description one can check directly (with some tedious computations) that the images of $\otimes(\xi,\eta)$ and $\otimes(\xi',\eta')$ in $\wedge^{p+q}(U)$ are equal. On the other hand, if one has repeated factors in either ξ or η and the corresponding object ξ' or η' is zero, then it is immediately clear that $\otimes(\xi,\eta)$ and $\otimes(\xi',\eta')$ in $\wedge^{p+q}(U)$ both zero and hence are equal.

PROPOSITION 3. *If $\theta \in \wedge^p(U)$ and $\omega \in \wedge^q(U)$, then we have $\theta\wedge\omega = (-1)^{pq}\,\omega\wedge\theta$.*

Proof. Using bilinearity we may immediately reduce this to the special case where

$$\theta \;=\; dx^{i_1}\wedge\;\cdots\;\wedge dx^{i_p} \quad , \quad \omega \;=\; dx^{j_1}\wedge\;\cdots\;\wedge dx^{j_q}\ .$$

In this case we have

$$\theta\wedge\omega \;=\; dx^{i_1}\wedge\;\cdots\;\wedge dx^{i_p}\wedge dx^{j_1}\wedge\;\cdots\;\wedge dx^{j_q} \quad , \quad \omega\wedge\theta \;=\; dx^{j_1}\wedge\;\cdots\;\wedge dx^{j_q}\wedge dx^{i_1}\wedge\;\cdots\;\wedge dx^{i_p}\ .$$

Therefore we need to investigate what happens if one rearranges the variables using some permutation.

If γ is an arbitrary permutation then γ is a product of transpositions, and therefore it follows that if one permutes variables by γ the effect on a basic monomial form is multiplication by $\mathrm{sgn}(\gamma)$. Therefore the proof of the formula in the proposition reduces to computing the sign of the permutation which takes the first p numbers in $\{1,\ \cdot\ , p+q\}$ to the last p numbers in order and takes the last q numbers to the first q numbers in order. It is an elementary combinatorial exercise to verify that the sign of this permutation is pq (*e.g.*, fix one of p or q and proceed by induction on the other$^{(\star)}$).■

The following property is also straightforward to verify$^{(\star)}$, and in fact it is a consequence of the analogous property for covariant tensors:

PROPOSITION 4. *If θ and ω are as above and $\lambda \in \wedge^r(U)$, then one has the associativity property $(\theta\wedge\omega)\wedge\lambda = \theta\wedge(\omega\wedge\lambda)$.*■

Exterior derivatives. We have already seen that there is a well-defined map $d : \wedge^0(U) \to \wedge^1(U)$ defined by taking exterior derivatives, and in fact for each p one can define an exterior derivative

$$d^p : \wedge^p(U) \;\longrightarrow\; \wedge^{p+1}(U)\ .$$

These maps are linear transformations of real vector spaces and are defined on monomials by the formula

$$d\big(\ g\,dx^{i_1}\wedge\;\cdots\;\wedge dx^{i_p}\ \big) \;=\; dg\wedge\ dx^{i_1}\wedge\;\cdots\;\wedge dx^{i_p}\ .$$

If we take $g = 1$ the preceding definition implies

$$d\big(\ dx^{i_1}\wedge\;\cdots\;\wedge dx^{i_p}\ \big) \;=\; 0\ .$$

One then has the following basic consequences of the definitions.

THEOREM 5. *The exterior derivative satisfies the following identities:*

(*i*) *If θ is a p-form then $d(\theta \wedge \omega) \;=\; (d\theta) \wedge \omega \;+\; (-1)^p \theta \wedge (d\omega)$.*

(*ii*) *For all λ we have $d(d\lambda) = 0$.*

Sketch of proof. In each case one can use linearity or bilinearity to reduce everything to the special case of forms that are monomials. For examples of this type it is a routine computational exercise to verify the identities described above$^{(\star)}$.■

Definition. A differential form ω is said to be *closed* if $d\omega = 0$ and exact if $\omega = d\lambda$ for some λ. The second part of the theorem implies that exact forms are closed. On the other hand, the 1-form

$$\frac{y\,dx - x\,dy}{x^2 + y^2}$$

on $\mathbb{R}^2 - \{0\}$ is closed but not exact.

Change of variables (pullbacks). The pullback construction on 1-forms extends naturally to forms of higher degree. Specifically, if V is open in $\mathbb{R}^m$ and $f : V \to U$ is smooth then there are real vector space homomorphisms $f^* : \wedge^p(U) \to \wedge^p(U)$ that are defined on monomials by the formula

$$f^*\big(\, g\, dx^{i_1} \wedge \;\cdots\; \wedge dx^{i_p} \,\big) \quad = \quad (g \circ f)\, df^{i_1} \wedge \;\cdots\; \wedge df^{i_p}$$

where f^i denotes the i^{th} coordinate function of f. If $p = 1$ this coincides with the previous definition.

The next result implies that the pullback construction preserves all the basic structure on exterior forms that we defined above and it has good naturality properties:

THEOREM 6. (*i*) *In the notation above we have $f^*(\theta \wedge \omega) = f^*\theta \wedge f^*\omega$ and $f^* \circ d\lambda = d \circ f^*\lambda$.*

(*ii*) *The pullback map for id_U is the identity on $\wedge^p(U)$, and if $h : W \to V$ is another smooth map, then $(f \circ h)^* = h^* \circ f^*$.*

(*iii*) *The pullback maps and exterior derivatives satisfy the compatibility relations $d \circ f^* = f^* \circ d$.*

Complete derivations of these results appear on pages 263–264 of Rudin$^{(\star)}$.■

The pullback also has the following basic compatibility property with respect to integrals:

CHANGE OF VARIABLES FOR INTEGRALS. *Let $\omega \in \wedge^p(U)$, let $f : V \to U$ be smooth, and let $\sigma : \Delta \to V$ be a smooth p-surface. Then integration of differential forms satisfies the following change of variables formula:*

$$\int_\Delta f^*\omega \quad = \quad \int_{f \circ \sigma} \omega$$

A derivation of this formula appears on pages 264–266 of Rudin$^{(\star)}$.■

Relation to classical vector analysis

We shall now explain how the basic constructions and main theorems of vector analysis can be expressed in terms of differential forms. For most of this section U will denote an open subset of $\mathbb{R}^3$.

Let $\mathbf{X}(U)$ be the Lie algebra of smooth vector fields on U. As a module over $\mathcal{C}^\infty(U)$ the space of vector fields is isomorphic to each of $\wedge^1(U)$ and $\wedge^2(U)$, and $\mathcal{C}^\infty(U)$ is isomorphic to $\wedge^3(U)$; recall that $\mathcal{C}^\infty(U) = \wedge^0(U)$ by definition. For our purposes it is important to give specific isomorphisms $\Phi^1 : \mathbf{X}(U) \to \wedge^1(U)$, $\Phi^2 : \mathbf{X}(U) \to \wedge^2(U)$, $\Phi^3 : \mathcal{C}^\infty(U) \to \wedge^3(U)$. A vector field will be viewed as a vector valued function $\mathbf{V} = (F, G, H)$ where each of F, G, H is a smooth real valued function on U.

$$\Phi_1(F,G,H) \quad = \quad F\,dx + G\,dy + H\,dz$$

$$\Phi_2(F,G,H) \quad = \quad F\,dy \wedge dx + \ G\,dz \wedge dx + H\,dx \wedge dy$$

$$\Phi_3(f) \quad = \quad f\,dx \wedge dy \wedge dx$$

We then have the following basic relationships:

$$(i) \quad \nabla f \quad = \quad \Phi_1^{-1}(df)$$

$$(ii) \quad \mathbf{curl}(\mathbf{V}) \quad = \quad \Phi_2^{-1} \circ d \circ \Phi_1(\mathbf{V})$$

$$(iii) \quad \mathbf{div}(\mathbf{V}) \quad = \quad \Phi_3^{-1} \circ d \circ \Phi_2(\mathbf{V})$$

Each of these is a routine computation$^{(\star)}$.

From this perspective the vector analysis identities

$$\mathbf{curl}(\nabla f) = 0 \quad , \quad \mathbf{div}\,\mathbf{curl}(\mathbf{V}) \quad = \quad 0$$

are equivalent to special cases of the more general relationship $d \circ d = 0$.

V.1 : Smooth singular chains

(Hatcher, §§ 2.1, 2.3; Conlon, § 8.2; Lee, Ch. 16)

We now need to introduce yet another way of computing the homology groups of an open subset of $\mathbb{R}^n$ for some n.

Let q be a nonnegative integer. In Unit II we defined a *singular q-simplex* in a topological space X to be a continuous mapping $T : \Delta_q \to X$, where Δ_q is the simplex in $\mathbb{R}^{q+1}$ whose vertices are the standard unit vectors; the group of singular q-chains $\mathbf{S}_q(X)$ was then defined to be the free abelian group on the set of singular q-simplices. The first step in this section is to is to define an analog of these groups involving *smooth* mappings if X is an open subset of $\mathbb{R}^n$ for some n.

Definition. Let q be a nonnegative integer, and let $\Lambda_q \subset \mathbb{R}^q$ be the q-simplex whose vertices are $\mathbf{0}$ and the standard unit vectors. Also, let U be an open subset of $\mathbb{R}^n$ for some $n \geq 0$. A *smooth singular q-simplex* in U is a continuous map $T : \Lambda_q \to U$ which is *smooth* — in other words, there is some open neighborhood W_T of Λ_q in $\mathbb{R}^q$ such that T extends to a map $W_T \to U$ which is smooth in the usual sense (the coordinate functions have continuous partial derivatives of all orders). The group of *smooth singular q-chains* $\mathbf{S}_q^{\mathrm{smooth}}(U)$ is the free abelian group on all smooth singular q-simplices in U.

There is an obvious natural relationship between the smooth and ordinary singular chain groups which is given by the standard affine isomorphism φ from Δ_q to Λ_q defined on vertices by $\varphi(\mathbf{e}_1) = \mathbf{0}$ and $\varphi(\mathbf{e}_i) = \mathbf{e}_{i-1}$ for all $i > 1$. Specifically, each smooth singular q-simplex $T : \Lambda_q \to U$ determines the continuous singular q-simplex $T \circ \varphi : \Delta_q \to U$. The resulting map of singular chain groups will be denoted by

$$\varphi^{\#} : \mathbf{S}_q^{\mathrm{smooth}}(U) \longrightarrow \mathbf{S}_q(U)$$

with subscripts or superscripts added if it is necessary to keep track of q or U.

One important feature of the ordinary singular chain groups is that they can be made into a chain complex, and it should not be surprising to learn that there is a compatible chain complex structure on the groups of smooth singular chains. We recall the definition of the chain complex structure on $\mathbf{S}_*(X)$ for a topological space X, starting with the preliminary constructions. If Δ_q is the standard q-simplex, then for each i such that $0 \leq i \leq q$ there is an i^{th} face map $\partial_i : \Delta_{q-1} \to \Delta_q$ sending the domain to the face of Δ_q opposite the vertex $\mathbf{e}_{i+1}$ with $\partial_i(\mathbf{e}_j) = \mathbf{e}_j$ if $j \leq i$ and $\partial_i(\mathbf{e}_j) = \mathbf{e}_{j+1}$ if $j \geq i+1$. Then each face map ∂_i defines function from singular q-simplices to singular $(q-1)$-simplices by the formula $\partial_i(T) = T \circ \partial_i$, and the formula

$$d_q = \sum_{i=0}^{q} (-1)^i \, \partial_i$$

defines a homomorphism from $\mathbf{S}_q(X)$ to $\mathbf{S}_{q-1}(X)$ with some important formal properties like $d_{q-1} \circ d_q = 0$.

For the analogous constructions on smooth singular chain groups, we first need compatible face maps on Λ_q. The simplest way to do this is to relabel the vertices of the latter as $\mathbf{0} = \mathbf{v}_0$ and $\mathbf{e}_i = \mathbf{v}_{i+1}$ for all i; then we may define ∂_i^{Λ} in the same way as ∂_i, the only difference being that we replace the vertices $\mathbf{e}_j$ for Δ_q by the vertices $\mathbf{v}_j$ for Λ_q.

We claim that if $T : \Lambda_q \to U$ is a smooth singular simplex then are all of the faces given by the composites $T \circ \partial_i^{\Lambda}$; this follows because each of maps ∂_i^{Λ} is an affine mapping and hence is smooth.

It follows immediately that the preceding constructions are compatible with the simplex isomorphisms φ constructed above, so that $\varphi^{\#} \circ \partial_i = \partial_i^{\Lambda} \circ \varphi^{\#}$, and if we define

$$d_q^{\mathrm{smooth}} : \mathbf{S}_q^{\mathrm{smooth}}(U) \longrightarrow \mathbf{S}_{q-1}^{\mathrm{smooth}}(U)$$

to be the sum of the terms $(-1)^i \partial_i^{\Lambda}$, then one has the following compatibility between smooth and singular chains.

PROPOSITION 1. *Let U be an open subset of $\mathbb{R}^n$ for some n, and let $\varphi^{\#} : \mathbf{S}_q^{\mathrm{smooth}}(U) \to \mathbf{S}_q(U)$ and d_*^{smooth} be the map given by the preceding constructions. Then the latter map makes $\mathbf{S}_*^{\mathrm{smooth}}(U)$ into a chain complex such that $\varphi^{\#}$ is a morphism of chain complexes.*

The assertion in the first sentence can be verified directly from the definitions, and the first assertion in the second sentence follows from the same sort of argument employed earlier in these notes. Finally, the fact that $\varphi^{\#}$ is a chain complex morphism is an immediate consequence of the assertion in the first sentence and the definitions of the differentials in the two chain complexes in terms of the maps ∂_i and ∂_i^{Λ}.■

We shall denote the homology of the complex of smooth singular chains by $H_*^{\mathrm{smooth}}(U)$ and call the associated groups the *smooth singular homology groups* of the open set $U \subset \mathbb{R}^n$. Later in this section we shall prove the following fundamentally important result.

ISOMORPHISM THEOREM. *For all open subsets $U \subset \mathbb{R}^n$, the associated homology morphism $\varphi_*^{\#}$ from the smooth singular homology groups $H_*^{\mathrm{smooth}}(U)$ to the ordinary singular homology groups $H_*(U)$.*

Functoriality properties

In order to prove the Isomorphism Theorem, we need to establish additional properties of smooth singular chain and homology groups that are similar to basic properties of ordinary singular chain and homology groups. The first of these is a basic naturality property:

PROPOSITION 2. *Let $U \subset \mathbb{R}^n$, (etc.) be as above, let $V \subset \mathbb{R}^m$ be open, and let $f : U \to V$ be a smooth mapping from U to V (the coordinates have continuous partial derivatives of all orders). Then there is a functorial chain map $f_{\#}^{\mathrm{smooth}} : \mathbf{S}_*^{\mathrm{smooth}}(U) \to \mathbf{S}_*^{\mathrm{smooth}}(V)$ such that $f_{\#}^{\mathrm{smooth}}$ maps a smooth singular q-simplex T to $f \circ T$ and we have the naturality property*

$$f_{\#} \circ \varphi^{\#} \quad = \quad \varphi^{\#} \circ f_{\#}^{\mathrm{smooth}}$$

where $f_{\#}$ is the corresponding map of smooth singular chains from $\mathbf{S}_*(U)$ to $\mathbf{S}_*(V)$.■

COROLLARY 3. *In the setting of the preceding result, one has functorial homology homomorphisms on smooth singular homology, and the maps $\varphi_*^{\#}$ define natural transformations from smooth singular homology to ordinary singular homology.*■

Combining this with the Isomorphism Theorem mentioned earlier, we see that the construction $\varphi_*^{\#}$ determines a natural isomorphism from smooth singular homology to ordinary singular homology for open subsets of Euclidean spaces.

Since we are already discussing functoriality, this is a good point to mention some properties of this sort which hold for differential forms but were not formulated in Section 0:

THEOREM 4. *Let $f : U \to V$ and $g : V \to W$ be smooth mappings of open subsets in Cartesian (Euclidean) spaces $\mathbb{R}^n$ where n need not be the same for any of U, V, W. Then the pullback construction on differential forms satisfies the identity $(g \circ f)^{\#} = f^{\#} \circ g^{\#}$. Furthermore, if f is the identity on U then $f^{\#}$ is the identity on $\wedge^*(U)$.*

The second of these is trivial, and the first is a direct consequence of the definitions and the Chain Rule for derivatives of composite maps.■

Our objective is to show that the natural map from smooth singular chains to ordinary chains

$$S_*^{\text{smooth}}(U) \longrightarrow S_*(U)$$

defines isomorphisms in homology and in cohomology with real coefficients if U is an arbitrary open subset of some $\mathbb{R}^n$.

It will be convenient to extend the definition of smooth singular chain complexes to arbitrary subsets of $\mathbb{R}^n$ for some n. Specifically, if $A \subset \mathbb{R}^n$ then the smooth singular chain complex $S_*^{\text{smooth}}(A)$ is defined so that each group $S_q(A)$ is free abelian on the set of continuous mappings $T : \Lambda_q \to A$ which extend to smooth mappings T' from some open neighborhood $W(T')$ of Λ_q to $\mathbb{R}^n$. If A is an open subset of $\mathbb{R}^n$, then this is equivalent to the original definition, for if we are given T' as above we can always find an open neighborhood V of Λ_q such that T' maps V into A.

Clearly the definitions of smooth and ordinary singular chains are similar, and in fact many properties of ordinary singular chain complexes extend directly to smooth singular chain complexes. The following two are particularly important:

(0) If A is a convex subset of $\mathbb{R}^n$ (which is not necessarily open), then the constant map defines an isomorphism from $H_q^{\text{smooth}}(A)$ to $H_q^{\text{smooth}}(\mathbb{R}^0)$ for all q; in particular, these groups vanish unless $q = 0$.

(1) If we are given two smooth maps $f, g : U \to V$ such that f and g are smoothly homotopic, then the chain maps from $S_*^{\text{smooth}}(U)$ to $S_*^{\text{smooth}}(V)$ determined by f and g are chain homotopic.

(2) The construction of barycentric subdivision chain maps $\beta : S_*(U) \to S_*(U)$ in Section I.2 of these notes, and the related chain homotopy from β to the identity, determine compatible mappings of the same type on smooth singular chain complexes.

The first two of these follow because the chain homotopy constructions from Section I.5 clearly send smooth chains to smooth chains. The proof of the final assertion has two parts. First, the barycentric subdivision chain map in Section I.2 takes singular chains in the images of the canonical mappings

$$S_*^{\text{smooth}}(W) \longrightarrow S_*(W)$$

into chains which also lie in the images of such mappings. However, the construction of the chain homotopy must be refined somewhat in order to ensure that it sends smooth chains to smooth chains. In order to construct such a refinement, one needs to know that the homology of $S_*^{\text{smooth}}(\Lambda_q)$ is isomorphic to the homology of a point (hence is zero in positive dimensions). The latter is true by Property (0).∎

As in the ordinary case, if $\mathcal{W}$ is an open covering of an open set $U \subset \mathbb{R}^n$, then one can define the complex $\mathcal{W}$-small singular chains

$$S_*^{\text{smooth},\mathcal{W}}(U)$$

generated by all smooth singular simplices whose images lie inside a single element of $\mathcal{W}$, and the argument for ordinary singular chains implies that the inclusion map

$$S_*^{\text{smooth},\mathcal{W}}(U) \longrightarrow S_*^{\mathcal{W}}(U)$$

defines isomorphisms in homology. The latter in turn implies that one has long exact Mayer-Vietoris sequences relating the smooth singular homology groups of U, V, $U \cap V$ and $U \cup V$, where U and V are open subsets of (the same) $\mathbb{R}^n$, and in fact one has a long commutative ladder diagram relating the Mayer-Vietoris sequences for (U, V) with smooth singular chains and ordinary singular chains.

The smooth and ordinary singular chain groups for $\mathbb{R}^0$ are identical, and therefore their smooth and ordinary singular homology groups are isomorphic under the canonical map from smooth to ordinary singular homology. By the discussion above, it follows that the canonical map

$$\varphi_*^U : S_*^{\text{smooth}}(U) \longrightarrow S_*(U)$$

is an isomorphism if U is a convex open subset of some $\mathbb{R}^n$. The next step is to extend the class of open sets for which φ_*^U is an isomorphism.

THEOREM 5. *The map φ_*^U is an isomorphism if U is a finite union of convex open subsets in $\mathbb{R}^n$.*

Proof. Let (C_k) be the the statement that φ_*^U is an isomorphism if U is a union of at most k convex open subsets. Then we know that (C_1) is true. Assume that (C_k) is true; we need to show that the latter implies (C_{k+1}).

The preceding statements about ladder diagrams and the Five Lemma imply the following useful principle: *If we know that φ_*^U, φ_*^V, and $\varphi_*^{U \cap V}$ are isomorphisms in all dimensions, then the same is true for $\varphi_*^{U \cup V}$.* — Suppose now that we have a finite sequence of convex open subsets $W_1, \cdots, W_{k+1}$, and take U and V to be $W_1 \cup \cdots \cup W_k$ and W_{k+1} respectively. Then we know that φ_*^U and φ_*^V are isomorphisms by the inductive hypotheses. Also, since

$$U \cap V \quad = \quad (W_1 \cap W_{k+1}) \cup \cdots \cup (W_k \cap W_{k+1})$$

and all intersections $W_i \cap W_j$ are convex, it follows from the induction hypothesis that $\varphi_*^{U \cap V}$ is an isomorphism in all dimensions. Therefore by the observation at the beginning of this paragraph we know that $\varphi_*^{U \cup V}$ is an isomorphism, which is what we needed in order to complete the inductive step.∎

To complete the proof that φ_*^U is an isomorphism for all U, we need the so-called *compact carrier properties* of singular homology. There are two versions of this result.

THEOREM 6. *Let X be a topological space, and let $u \in H_q(X)$. Then there is a compact subset $K \subset X$ such that u lies in the image of the canonical map from $H_q(K)$ to $H_q(X)$. Furthermore, if K is a compact subset of X, and v and w are classes in $H_q(K)$ whose images in $H_q(X)$ are equal, then there is a compact subset L such that $K \subset L \subset X$ such that the images of v and w are equal in $H_q(L)$.*

Proof. Choose a singular chain $\sum_i \, n_i \, T_i$ representing u, where each T_i is a continuous mapping $\Delta_q \to X$. If K is the union of the images $T_i[\Delta_q]$, then K is compact, and it follows that u lies in the image of $H_q(K)$ (because the chain lies in the subcomplex $S_*(K) \subset S_*(X)$.

To prove the second assertion in the proposition, note that by additivity it suffices to prove this when $w = 0$. Once again choose a representative singular chain $\sum_i \, n_i \, T_i$ for v; since the image of v in $H_q(X)$ is a boundary, there is a $(q+1)$-chain $\sum_j \, m_j \, U_j$ on X whose boundary is $\sum_i \, n_i \, T_i$. Let

L be the union of K and the compact sets $U_j[\Delta_{q+1}]$; then L is compact and it follows immediately that v maps to zero in $H_q(L)$.■

We shall need a variant of the preceding result.

THEOREM 6. *Let U be an open subset of some $\mathbb{R}^n$, and let $u \in H_q^{\mathrm{CAT}}(U)$, where CAT denotes either ordinary singular homology or smooth singular homology. Then there is a finite union of convex open subsets $V \subset U$ such that u lies in the image of the canonical map from $H_q^{\mathrm{CAT}}(V)$ to $H_q^{\mathrm{CAT}}(U)$. Furthermore, if V is a finite union of convex open subsets of U, and v and w are classes in $H_q^{\mathrm{CAT}}(V)$ whose images in $H_q^{\mathrm{CAT}}(U)$ are equal, then there is a finite union of convex open subsets W such that $V \subset W \subset U$ such that the images of v and w are equal in $H_q^{\mathrm{CAT}}(W)$.*

Proof. The argument is similar, so we shall merely indicate the necessary changes. We adopt all the notation from the preceding discussion.

For the first assertion, by compactness we know that there is a finite union of convex open subsets V such that $K \subset V \subset U$, and it follows that u lies in the image of the homology of V. For the second assertion, take W to be the union of V and finitely many convex open subsets whose union contains L. It then follows that v maps to zero in the homology of W.■

We can now prove the following general result.

THEOREM 7. *The map φ_*^U is an isomorphism for arbitrary open subsets of some $\mathbb{R}^n$.*

Proof. If $u \in H_q(U)$, then we know there is some finite union of convex open subsets V such that $u = i_*(u_1)$, where $i : V \subset U$ is inclusion. By our previous results we know that $u_1 = \varphi_*^V(u_2)$ for some $u_2 \in H_q^{\mathrm{smooth}}(V)$, and since $i_* \circ \varphi_*^V = \varphi_*^U \circ i_*$, it follows that $u = \varphi_*^U i_*(u_2)$, so that φ_*^U is onto.

To show that φ_*^U is 1–1, suppose that v lies in its kernel. By the previous results we know that v lies in the image of $H_q^{\mathrm{smooth}}(V)$; suppose that v_1 maps to v. Then it follows that $v_2 = \varphi_*^V(v_1) \in H_q(V)$ maps to zero in $H_q(U)$, so that there is a finite union of convex open subsets W such that $V \subset W$ and v_2 maps to zero in $H_q(W)$. If $j : V \to W$ is inclusion, then it follows that $j_*(v_1)$ lies in the kernel of φ_*^W; however, we know that the latter map is 1–1 and therefore it follows that $j_*(v_1) = 0$. Since the image of the latter element in $H_*^{\mathrm{smooth}}(U)$ is equal to v, it follows that $v = 0$ and hence φ_*^U is 1–1, which is what we wanted to prove.■

Smooth singular cochains

As in Unit IV, we can dualize the construction of smooth singular chains to obtain smooth singular cochain groups for an open subset $U \subset \mathbb{R}^n$. Specifically, if M is an abelian group then the smooth singular cochain complex is defined by

$$S^*_{\mathrm{smooth}}(U; M) \quad = \quad \mathrm{Hom}\left(S_*^{\mathrm{smooth}}(U), M\right)$$

with the coboundary δ^* given by $\mathrm{Hom}(d_*, M)$.

If we are given a smooth map of open subsets in Euclidean spaces $f : U \to V$ and its associated map of smooth singular chain complexes $f_{\#}$, then we have maps of singular cochain complexes

$$f^{\#} \;=\; \mathrm{Hom}(f^{\#}, M) : S^*_{\mathrm{smooth}}(V; M) \to S^*_{\mathrm{smooth}}(U; M)$$

and morphisms of cohomology groups $f^* : H^*_{\rm smooth}(V;M) \to H^*_{\rm smooth}(U;M)$ which are contravariantly functorial with respect to smooth mappings. Furthermore, for open subsets in Euclidean spaces the canonical natural transformation from $S^{\rm smooth}_*(U)$ to $S_*(U)$ defines natural transformations of cochain complexes

$$\varphi^{\#\#} : S^*(U;M) \longrightarrow S^*_{\rm smooth}(U;M)$$

and cohomology groups $H^*(U;M) \to H^*_{\rm smooth}(U;M)$ which are natural with respect to smooth maps.

Theorem 7 and the weak Universal Coefficient Theorem of Unit IV immediately yields the following result for cohomology with field coefficients:

THEOREM 8. *If $\mathbb{F}$ is a field and U is an arbitrary open subset of $\mathbb{R}^n$, then the map φ^*_U : $H^*(U;\mathbb{F}) \longrightarrow H^*_{\rm smooth}(U;\mathbb{F})$ is an isomorphism of real vector spaces.*■

V.2 : Generalized Stokes' Formula

(Conlon, §§ 2.6, 8.1–8.2; Lee, Ch. 14)

At the end of Section 0 we discussed a far-reaching extension of the classical theorems of vector analysis (including the Fundamental Theorem of Calculus) to higher dimensions. In this section we shall formulate a version of this generaliztion which plays the key role in relating smooth singular cochains to differential forms.

Integration over smooth singular chains

If U is an open subset of $\mathbb{R}^n$ and $T\Lambda_q \to U$ is a smooth singular q-simplex, then the basic integration formula in Section V.0 provides a way of defining an integral $\int_T \omega$ if $\omega \in \wedge^q(U)$. There is a natural extension of this to singular chains; if $\mathbf{c}$ is the smooth singular chain $\sum_i n_i T_i$ where the n_i are integers, then since the group of smooth singular q-chains is free abelian on the smooth singular q-simplices the following is well defined:

$$\int_{\mathbf{c}} \omega \quad = \quad \sum_i n_i \int_{T_i} \omega$$

This definition has the following invariance property with respect to smooth mappings $f : U \to V$.

PROPOSITION 1. *Let $\mathbf{c} \in S^{\rm smooth}_q(U)$, where U is above, let $f : U \to V$ be smooth and let $\omega \in \wedge^q(V)$. Then we have*

$$\int_{f^{\rm smooth}_{\#}(\mathbf{c})} \omega \quad = \quad \int_{\mathbf{c}} f^{\#}\omega \ .$$

This follows immediately from the definition of integrals and the Chain Rule.■

The combinatorial form of the **Generalized Stokes' Formula** is a statement about integration of forms over smooth singular chains.

THEOREM 2. (Generalized Stokes' Formula, combinatorial version) *Let* $\mathbf{c}$, U, ω...(*etc.*) *be as above. Then we have*

$$\int_{d\,\mathbf{c}} \omega \;=\; \int_{\mathbf{c}} d\,\omega\ .$$

Full proofs of this result appear on pages 251–253 of Conlon and also on pages 272–275 of Rudin, *Principles of Mathematical Analysis* ($3^{\rm rd}$ Ed.)$^{(\star)}$. Here is an outline of the basic steps: First of all, by additivity it is enough to prove the result when $\mathbf{c}$ is given by a smooth singular simplex T. Next, by Proposition 1 and the identity $f^{\#}\circ d \;=\; d\circ f^{\#}$, we know that it suffices to prove the result when T is the universal singular simplex $\mathbf{1}_q$ defined by the inclusion of Λ_q — the simplex in $\mathbb{R}^q$ whose vertices are $\mathbf{0}$ and the unit vectors — into some small open neighborhood W_0 of Λ_q. In this case the integrals reduce to ordinary integrals in $\mathbb{R}^q$. We can reduce the proof even further as follows: Let $\theta_i \in \wedge^{q-1}(W_0)$ be the basic $(q-1)$-form $dx^{i_1} \wedge \;\cdots\; dx^{i_{q-1}}$, where $i_1 < \;\cdots\; < i_{q-1}$ runs over all elements of $\{1, \;\cdots\;, q\}$ **except** i. By additivity it will suffice to prove the theorem for $(q-1)$-forms expressible as $g\,\theta_i$, where g is a smooth function on W_0. Yet another change of variables argument shows that it suffices to prove the result for $(q-1)$-forms expressible as $g\,dx^2 \wedge \;\cdots\; \wedge dx^q$. Now the exterior derivative of the latter form is equal to

$$\frac{\partial g}{\partial x^1} \cdot dx^1 \wedge \;\cdots\; dx^q$$

so the proof reduces to evaluating the integral of the left hand factor in this expression over Λ_q, and this is done by viewing this multiple integral as an interated integral via Fubini's Theorem (see Rudin, *Real and complex analysis* or almost any text discussing Lebesgue integration) and applying the Fundamental Theorem of Calculus.■

RELATION TO CLASSICAL VECTOR ANALYSIS. The identifications in $(i)-(iii)$ lead to a general statement that includes the following three basic results:

(1) The standard path independence result stating that the line integral $\int \nabla f \cdot d\mathbf{x}$ is equal to f(final point on curve) $-$ f(initial point on curve).

(2) Stokes' Theorem (*note the spelling!!*) relating line and surface integrals.

(3) The so-called Gauss or Divergence Theorem relating surface and volume integrals.

In each case the result can be stated in terms of differential forms and p-surfaces (where $p = 1, 2, 3$) as follows: *If we are given a p-surface σ that has a reasonable notion of boundary $\partial\sigma$ such that $\partial\sigma$ is somehow a sum of $(p-1)$-surfaces with coefficients of ± 1, then*

$$\int_{\partial\sigma} \omega \;=\; \int_{\sigma} d\omega$$

for all $(p-1)$-forms ω.

In all cases the relationship to the Generalized Stokes' Formula depends upon the existence of *piecewise smooth triangulations* for the domains in which the various integrals are defined. More precisely, these are families of mappings T_α from the standard simplices Λ_q satisfying the following conditions:

(a) The union of the images of the simplices is the entire domain of integration, and the intersection of two images is a common face.

(b) Each map T_α is smooth and 1–1, and the derivative matrix at each point (*i.e.*, the matrix of partial derivatives of the coordinate functions of T_α) normally has rank $q(\alpha)$; in some cases these statements can be weakened slightly to allow some irregular behavior on the boundaries.

(c) The structure described above induces a similar structure on the boundary of the domain of integration.

(d) The sum of the integrals with respect to the mappings T_α are the standard notion of integral for the domain under consideration, and likewise for the boundary.

One way of restating the final condition is to say that if one forms a triangulating chain for the domain of integration by adding the symbols $\pm T_\alpha$ formally , then the algebraic boundary of this chain (in the sense of singular homology) will be a triangulating chain for the boundary of the domain. Standard examples in multivariable calculus amount to saying that such a condition does not hold for a smooth bounded surface in $\mathbb{R}^3$ corresponding to a Möbius strip.

Results (1)–(3) are special cases of more general result which hold in all finite dimensions. Unfortunately, precise formulations of such generalizations require more background than we have developed (mainly from [MunkresEDT]), so we shall not try to state such results explicitly here.

V.3 : Definition and properties of de Rham cohomology

(Hatcher, §§ 2.1, 2.3, 3.1; Conlon, §§ 2.6, 8.1, 8.3–8.5; Lee, Ch. 15)

Let U be an open subset of $\mathbb{R}^n$ for some n. Since the exterior derivative on $\wedge^p(U)$ satisfies $d \circ d = 0$, it follows that $(\wedge^*(U), d^*)$ is a cochain complex, which we shall call the **de Rham (cochain) complex**.

Definition. The **de Rham cohomology groups** $H^q_{\mathrm{DR}}(U)$ are the cohomology groups of the de Rham complex of differential forms.

The Generalized Stokes' Formula in Theorem 2.2 implies that integration of differential forms defines a morphism J of chain complexes from $\wedge^*(U)$ to $S^*(U;\mathbb{R})$, where U is an arbitrary open subset of some Euclidean space. The aim of this section and the next is to show that the associated cohomology map $[J]$ defines an isomorphism from $H^*_{\mathrm{DR}}(U)$ to $H^*_{\mathrm{smooth}}(U;\mathbb{R})$; by the results of the preceding section, it will also follow that the de Rham cohomology groups are isomorphic to the ordinary singular cohomology groups $H^*(U;\mathbb{R})$. In order to prove that $[J]$ is an isomorphism, we need to show that the de Rham cohomology groups $H^*_{\mathrm{DR}}(U)$ satisfy analogs of certain formal properties that hold for (smooth) singular cohomology.

One of these properties is a homotopy invariance principle, and the other is a Mayer-Vietoris sequence. Extremely detailed treatments of these results are given in Conlon, so at several points we shall be rather sketchy.

The following abstract result will be helpful in proving homotopy invariance. There are obvious analogs for other subcategories of topological spaces and continuous mappings, and also for covariant functors.

LEMMA 1. *Let T be a contravariant functor defined on the category of open subsets of $\mathbb{R}^n$ and smooth mappings. Then the following are equivalent:*

(1) *If f and g are smoothly homotopic mappings from U to V, then $T(f) = T(g)$.*

(2) *If U is an arbitrary open subset of $\mathbb{R}^n$ and $i_t : U \to U \times \mathbb{R}$ is the map sending u to (u,t), then $T(i_0) = T(i_1)$.*

Proof. (1) $\Longrightarrow$ (2). The mappings i_0 and i_1 are smoothly homotopic, and the inclusion map defines a homotopy from $U \times (-\varepsilon, 1+\varepsilon)$ to $U \times \mathbb{R}$.

(2) $\Longrightarrow$ (1). Suppose that we are given a smooth homotopy $H : U \times (-\varepsilon, 1+\varepsilon) \to V$. Standard results from 205C imply that we can assume the homotopy is "constant" on some sets of the form $(-\varepsilon, \eta) \times U$ and $(1-\eta, 1+\varepsilon) \times U$ for a suitably small positive number η. One can then use this property to extend H to a smooth map on $U \times \mathbf{R}$ that is "constant" on $(-\infty, \eta) \times U$ and $(1-\eta, \infty) \times U$. By the definition of a homotopy we have $H \circ i_1 = g$ and $H \circ i_0 = f$. If we apply the assumption in (1) we then obtain

$$T(g) \quad = \quad T(i_1) \circ T(H) \quad = \quad T(i_0) \circ T(H) \quad = \quad T(f)$$

which is what we wanted.■

A simple decomposition principle for differential forms on a cylindrical open set of the form $U \times \mathbb{R}$ will be useful. If U is open in $\mathbb{R}^n$ and I denotes the k-element sequence $i_1 < \cdots < i_k$, we shall write

$$\xi_I \quad = \quad dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and say that such a form is a *standard basic monomial k-forms* on U. Note that the wedge of two standard basic monomials $\xi_J \wedge \xi_I$ is either zero or ± 1 times a standard basic monomial, depending upon whether or not the sequences J and I have any common wedge factors.

PROPOSITION 2. *Every k-form on U is uniquely expressible as a sum*

$$\sum_I f_I(x,t)\, dt \wedge \xi_I \quad + \quad \sum_J g_J(x,t)\, \xi_J$$

where the index I runs over all sequences $0 < i_1 < \cdots < i_{k-1} \leq n$, the index J runs over all sequences $0 < j_1 < \cdots < j_k \leq n$, and f_I, g_J are smooth functions on $U \times \mathbb{R}$.■

We then have the following basic result.

THEOREM 3. *If U is an open subset of some $\mathbb{R}^n$ and $i_t : U \to U \times \mathbb{R}$ is the map $i_t(x) = (x,t)$, then the associated maps of differential forms $i_0^{\#}, i_1^{\#} : \wedge^*(U \times \mathbb{R}) \to \wedge^*(U)$ are chain homotopic.*

In this example the chain homotopy is frequently called a *parametrix*.

COROLLARY 4. *In the setting above the maps i_0^* and i_1^* from $H^*_{\mathrm{DR}}(U \times \mathbb{R})$ to $H^*_{\mathrm{DR}}(U)$ are equal.*■

Proof of Theorem 3. The mappings $P^q : \wedge^q(U \times \mathbb{R}) \to \wedge^{q-1}(U)$ are defined as follows. If we write a q-form over $U \times \mathbb{R}$ as a sum of terms $\alpha_I = f_I(x,t)\, dt \wedge \xi_I$ and $\beta_J = g_J(x,t)\, \xi_J$ using the lemma above, then we set $P^q(\beta_J) = 0$ and

$$P^q(\alpha_I) \quad = \quad \left(\int_0^1 f_I(x,u)\, du \right) \cdot \xi_I \ ;$$

we can then extend the definition to an arbitrary form, which is expressible as a sum of such terms, by additivity.

We must now compare the values of $dP + Pd$ and $i_1^\# - i_0^\#$ on the generating forms α_I and β_J described above. It follows immediately that $i_1^\#(\alpha_I) - i_0^\#(\alpha_I) = 0$ and

$$i_1^\#(\beta_J) \;-\; i_0^\#(\beta_J) \;\;=\;\; [g(x,1) \;-\; g(x,0)]\,\beta_J \;.$$

Next, we have $d\circ P(\beta_J) = d(0) = 0$ and

$$d\circ P(\alpha_I) \;\;=\;\; d\left(\int_0^1 f_I(x,u)du)\right)\cdot \xi_I \;\;=$$

$$\sum_j \;\left(\int_0^1 \frac{\partial f_I}{\partial x^j}(x,u)\,du\right) \wedge dx^j \wedge \omega_I \;.$$

Similarly, we have

$$P\circ d(\alpha_I) \;\;=\;\; P\left(\sum_j \frac{\partial f_I}{\partial x^j}\,dx^j \wedge dt \wedge \xi_I \;+\; \frac{\partial f_I}{\partial t}\,dt \wedge dt \wedge \xi_I\right)$$

in which the final summand vanishes because $dt \wedge dt = 0$. If we apply the definition of P to the nontrivial summation on the right hand side of the displayed equation and use the identity $dx^j \wedge dt = -dt \wedge dx^j$, we see that the given expression is equal to $-d\circ P(\alpha_I)$; this shows that the values of both $dP + Pd$ and $i_1^\# - i_0^\#$ on α_I are zero. It remains to compute $P\circ d(\beta_J)$ and verify that it is equal to $i_1^\#(\beta_J) - i_0^\#(\beta_J)$. However, by definition we have

$$P\circ d(g_J\,\xi_J) \;\;=\;\; P\left(\sum_i \frac{\partial g_J}{\partial x^i}\,dx^i \wedge \xi_J \;+\; \frac{\partial g_J}{\partial t}\,dt \wedge \xi_J\right)$$

and in this case P maps the summation over i into zero because each form $dx^i \wedge \xi_J$ is either zero or ± 1 times a standard basic monomial, depending on whether or not dx^i appears as a factor of ξ_J. Thus the right hand side collapses to the final term and is given by

$$P\left(\frac{\partial g_J}{\partial t}\,dt \wedge \xi_J\right) \;\;=\;\; \left(\int_0^1 \frac{\partial g_J}{\partial u}(x,u)\,du\right)\xi_J \;\;=$$

$$[\,g(x,1) \;-\; g(x,0)\,]\;\xi_J$$

which is equal to the formula for $i_1^\#(\beta_J) - i_0^\#(\beta_J)$ which we described at the beginning of the argument.∎

COROLLARY 5. *If U is a convex open subset of some $\mathbb{R}^n$, then $H^q_{\mathrm{DR}}(U)$ is isomorphic to $\mathbb{R}$ if $q = 0$ and is trivial otherwise.*

This follows because the constant map from U to $\mathbb{R}^0$ is a smooth homotopy equivalence if U is convex, so that the de Rham cohomology groups of U are isomorphic to the de Rham cohomology groups of $\mathbb{R}^0$, and by construction the latter are isomorphic to the groups described in the statement of the Corollary.■

COROLLARY 6. (Poincaré Lemma) *Let U be a convex open subset of some $\mathbb{R}^n$ and let $q > 0$. The a differential q-form ω on U is closed ($d\omega = 0$) if and only if it is exact ($\omega = d\theta$ for some θ).*■

Both of the preceding also hold if we merely assume that U is star-shaped with respect to some point $\mathbf{v}$ (*i.e.*, if $\mathbf{x} \in U$, then the closed line segment joining $\mathbf{x}$ and $\mathbf{v}$ is contained in U), for in this case the constant map is again a smooth homotopy equivalence.■

The Mayer-Vietoris sequence

Here is the main result:

THEOREM 7. *Let U and V be open subsets of $\mathbb{R}^n$. Then there is a long exact Mayer-Vietoris sequence in de Rham cohomology*

$$\cdots \to H^{q-1}_{\mathrm{DR}}(U \cap V) \to H^q_{\mathrm{DR}}(U \cup V) \to H^q_{\mathrm{DR}}(U) \oplus H^q_{\mathrm{DR}}(V) \to H^q_{\mathrm{DR}}(U \cap V) \to H^{q+1}_{\mathrm{DR}}(U \cup V) \to \cdots$$

and a commutative ladder diagram relating the long exact Mayer-Vietoris sequences for $\{U, V\}$ in de Rham cohomology and smooth singular cohomology with real coefficients.

Proof. The existence of the Mayer-Vietoris sequence will follow if we can show that there is a short exact sequence of chain complexes

$$0 \to \wedge^*(U \cup V) \longrightarrow \wedge^*(U) \oplus \wedge^*(V) \longrightarrow \wedge^*(U \cap V) \to 0$$

where the map from $\wedge^*(U \cup V)$ is given on the first factor by the $i_U^\#$ (where i_U denotes inclusion) and on the second factor by $-i_V^\#$, and the map into $\wedge^*(U \cap V)$ is given by the maps $j_U^\#$ and $j_V^\#$ defined by inclusion of $U \cap V$ into U and V.

The exactness of this sequence at all points except $\wedge^*(U \cap V)$ follows immediately. Therefore the only thing to prove is that the map to $\wedge^*(U \cap V)$ is surjective. This turns out to be less trivial than one might first expect (in contrast to singular cochains, a differential form on $U \cap V$ need not extend to either U or V), but it can be done using smooth partitions of unity. Specifically, let $\{\varphi_U, \varphi_V\}$ be a smooth partition of unity subordinate to the open covering $\{U, V\}$ of $U \cup V$, and let $\omega \in \wedge^p(U \cap V)$. Consider the forms $\varphi_U \cdot \omega$ and $\varphi_V \cdot \omega$ on $U \cap V$. By definition of a partition of unity there are open subsets $U_0 \subset U$ and $V_0 \subset V$ whose closures in $U \cup V$ are contained in U and V respectively, and such that φ_U and φ_V are zero off the closures of U_0 and V_0. This means that we can define a smooth form θ_U on U such that

$$\theta_U | U \cap V \quad = \quad \varphi_U \cdot \omega \ , \qquad \theta_U | U - \overline{U_0}$$

because both restrict to zero on $U \cap V - \overline{U_0}$. The same reasoning also yields a similar form θ_V on V, and it follows that

$$(\theta_U, \theta_V) \quad \in \quad \wedge^p(U) \ \oplus \ \wedge^p(V)$$

maps to $\omega \in \wedge^p(U \cap V)$. Additional details are given in Conlon (specifically, the last four lines of the proof for Lemma 8.5.1 on page 267).

The existence of the commutative ladder follows because the Generalized Stokes' Formula defines morphisms from the objects in the de Rham short exact sequence into the following analog for smooth singular cochains:

$$0 \to S^*_{\mathrm{smooth},\mathcal{U}}(U \cup V) \longrightarrow S^*_{\mathrm{smooth}}(U) \oplus S^*_{\mathrm{smooth}}(V) \longrightarrow S^*_{\mathrm{smooth}}(U \cap V) \to 0$$

The first term in this sequence denotes the cochains for the complex of $\mathcal{U}$-small chains on $U \cup V$, where $\mathcal{U}$ denotes the open covering $\{U, V\}$.

Since the displayed short exact sequence yields the long exact Mayer-Vietoris sequence for (smooth) singular cohomology, the statement about a commutative ladder in the theorem follows.■

V.4 : De Rham's Theorem

(Conlon, § 8.9; Lee, Chs. 15–16)

The results of the preceding section show that the natural map $[J] : H^*_{\mathrm{DR}}(U) \to H^*_{\mathrm{smooth}}(U; \mathbb{R})$ is an isomorphism if U is a convex open subset of some Euclidean space, and if we compose this with the isomorphism between smooth and ordinary singular cohomology we obtain an isomorphism from the de Rham cohomology of U to the ordinary singular cohomology of U with real coefficients. The aim of this section is to show that both $[J]$ and its composite with the inverse map from smooth to ordinary cohomology is an isomorphism for an arbitrary open subset of $\mathbb{R}^n$. As in Section II.2, an important step in this argument is to prove the result for open sets which are expressible as finite unions of convex open subsets of $\mathbb{R}^n$.

PROPOSITION 1. *If U is an open subset of $\mathbb{R}^n$ which is expressible as a finite union of convex open subsets, then the natural map from $H^*_{\mathrm{DR}}(U)$ to $H^*_{\mathrm{smooth}}(U; \mathbb{R})$ and the associated natural map to $H^*(U; \mathbb{R})$ are isomorphisms.*

Proof. If W is an open subset in $\mathbb{R}^n$ we shall let ψ^W denote the natural map from de Rham to singular cohomology. If we combine the Mayer-Vietoris sequence of the preceding section with the considerations of Section II.2, we obtain the following important principle:

> *If $W = U \cup V$ and the mappings ψ^U, ψ^V and $\psi(U \cap V)$ are isomorphisms, then $\psi^{U \cup V}$ is also an isomorphism.*

Since we know that ψ^V is an isomorphism if V is a convex open subset, we may prove the proposition by induction on the number of convex open subsets in the presentation $W = V_1 \cup \cdots \cup V_k$ using the same sorts of ideas employed in Section II.2 to prove a corresponding result for the map relating smooth and ordinary singular homology.■

Extension to arbitrary open sets

Most open subsets of $\mathbb{R}^n$ are not expressible as finite unions of convex open subsets, so we still need some method for extracting the general case. The starting point is the following observation, which implies that an open set is a *locally finite* union of convex open subsets.

THEOREM 2. *If U is an open subset of $\mathbb{R}^n$, then U is a union of open subsets W_n indexed by the positive integers such that the following hold:*

(1) *Each W_n is a union of finitely many convex open subsets.*

(2) *If $|m-n| \geq 3$, then $W_n \cap W_m$ is empty.*

Proof. Results from 205C imply that U can be expressed as an increasing union of compact subsets K_n such that K_n is contained in the interior of K_{n+1} and K_1 has a nonempty interior$^{(\star)}$. Define $A_n = K_n - \mathbf{Int}(K_{n-1})$, where K_{-1} is the empty set; it follows that A_n is compact. Let V_n be the open subset $\mathbf{Int}(K_{n+1}) - K_{n-1}$. By construction we know that V_n contains A_n and $V_n \cap V_m$ is empty if $|n-m| \geq 3$. Clearly there is an open covering of A_n by convex open subsets which are contained in V_n, and this open covering has a finite subcovering; the union of this finite family of convex open sets is the open set W_n that we want; by construction we have $A_n \subset W_n$, and since $U = \cup_n A_n$ we also have $U = \cup_n W_n$. Furthermore, since $W_n \subset V_n$, and $V_n \cap V_m$ is empty if $|n-m| \geq 3$, it follows that $W_n \cap W_m$ is also empty if $|n-m| \geq 3$.■

We shall also need the following result:

PROPOSITION 3. *Suppose that we are given an open subset U in $\mathbb{R}^n$ which is expressible as a countable union of pairwise disjoint subset U_k. If the map from de Rham cohomology to singular cohomology is an isomorphism for each U_k, then it is also an isomorphism for U.*

Proof. By construction the cochain and differential forms mappings determined by the inclusions $i_k : U_k \to U$ define morphisms from $\wedge^*(U)$ to the cartesian product $\Pi_k \wedge^*(U_k)$ and from $S^*_{\text{smooth}}(U)$ to $\Pi_k\ S^*_{\text{smooth}}(U_k)$. We claim that these maps are isomorphisms. In the case of differential forms, this follows because an indexed set of p-forms $\omega_k \in \wedge^p(U_k)$ determine a unique form on U (existence follows because the subsets are pairwise disjoint), and in the case of singular cochains it follows because every singular chain is uniquely expressible as a sum $\sum_k\ c_k$, where c_k is a singular chain on U_k and all but finitely many c_k's are zero (since the image of a singular simplex $T : \Delta_q \to U$ is pathwise connected and the open sets U_k are pairwise disjoint, there is a unique m such that the image of T is contained in U_m).

If we are given an abstract family of cochain complexes C_k then it is straightforward to verify that there is a canonical homomorphism

$$H^*\left(\prod_k\ C_k\right) \longrightarrow \prod_k\ H^*(C_k)$$

defined by the projection maps

$$\pi_j : \prod_k\ C_k \longrightarrow C_j$$

and that this mapping is an isomorphism. Furthermore, it is natural with respect to families of cochain complex mappings $f_k : C_k \to E_k$.

The proposition follows by combining the observations in the preceding two paragraphs$^{(\star)}$.■

We are now ready to prove the main result, which G. de Rham (1903–1990) first proved in 1931:

THEOREM 4. (de Rham's Theorem.) *The natural maps from de Rham cohomology to smooth and ordinary singular cohomology are isomorphisms for every open subset U in an arbitrary $\mathbb{R}^n$.*

Proof. Express U as a countable union of open subset W_n as in the discussion above, and for $k = 0, 1, 2$ let $U_k = \cup_m \; W_{3m+k}$. As noted in the definition of the open sets W_j, the open sets W_{3m+k} are pairwise disjoint. Therefore by the preceding proposition and the first result of this section we know that the natural maps from de Rham cohomology to singular cohomology are isomorphisms for the open sets U_k.

We next show that the natural map(s) must define isomorphisms for $U_0 \cup U_1$. By the highlighted statement in the proof of the first proposition in this section, it will suffice to show that the same holds for $U_0 \cap U_1$. However, the latter is the union of the pairwise disjoint open sets $W_{3m} \cap W_{3m+1}$, and each of the latter is a union of finitely many convex open subsets. Therefore by the preceding proposition and the first result of this section we know that the natural maps from de Rham to singular cohomology are isomorphisms for $U_0 \cap U_1$ and hence also for $U^* = U_0 \cup U_1$.

Clearly we would like to proceed similarly to show that we have isomorphisms from de Rham to singular cohomology for $U = U_2 \cup U^*$, and as before it will suffice to show that we have isomorphisms for $U_2 \cap U^*$. But $U_2 \cap U^* = (U_2 \cap U_0) \cup (U_2 \cap U_1)$. By the preceding paragraph we know that the maps from de Rham to singular cohomology are isomorphisms for $U_0 \cap U_1$, and the same considerations show that the corresponding maps are isomorphisms for $U_0 \cap U_2$ and $U_1 \cap U_2$. Therefore we have reduced the proof of de Rham's Theorem to checking that there are isomorphisms from de Rham to singular cohomology for the open set $U_0 \cap U_1 \cap U_2$. The latter is a union of open sets expressible as $W_i \cap W_j \cap W_k$ for suitable positive integers i, j, k which are distinct. The only way such an intersection can be nonempty is if the three integers i, j, k are *consecutive* (otherwise the distance between two of them is at least 3). Therefore, if we let

$$S_m \quad = \bigcup_{0 \leq k \leq 2} W_{3m-k} \cap W_{3m+1-k} \cap W_{3m+2-k}$$

it follows that S_m is a finite union of convex open sets, the union of the open sets S_m is equal to $U_0 \cap U_1 \cap U_2$, and if $m \neq p$ then $S_m \cap S_p$ is empty (since the first is contained in W_{3m} and the second is contained in the disjoint subset W_{3p}). By the first result of this section we know that the maps from de Rham to singular cohomology define isomorphisms for each of the open sets S_m, and it follows from the immediately preceding proposition that we have isomorphisms from de Rham to singular cohomology for $\cup_m \; S_m \; = \; U_o \cap U_1 \cap U_2$. As noted before, this implies that the corresponding maps also define isomorphisms for U.■

Some applications

In Section 6 we shall use de Rham's Theorem to generalize results multivariable calculus on path independence for line integrals in open subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$. For the time being we shall limit ourselves to verifying another result which sometimes appears in multivariable calculus texts.

PROPOSITION 5. *Suppose that $U \subset \mathbb{R}^3$ is a contractible open set and $\mathbf{F}$ is a smooth vector field on U whose divergence $\nabla \cdot \mathbf{F}$ is zero. Then $\mathbf{F} = \nabla \times \mathbf{P}$ for some vector field $\mathbf{P}$ on U.*

Proof. Given $\mathbf{F} = (F_1, F_2, F_3)$ as in the statement of the proposition, let $\theta_{\mathbf{F}}$ be the 2-form

$$F_1 \, dx_2 \wedge dx_3 \; + \; F_2 \, dx_3 \wedge dx_1 \; + \; F_3 \, dx_1 \wedge dx_2$$

and note that $d\theta_{\mathbf{F}} = (\nabla \cdot \mathbf{F}) \, dx_1 \wedge dx_2 \wedge dx_3$. Therefore the divergence condition translates into $d\theta_{\mathbf{F}} = 0$. Since $H^1_{\mathrm{DR}}(U) \cong H^2(U; \mathbb{R})$ by de Rham's Theorem and the latter is trivial by contractibility, it follows that $\theta_{\mathbf{F}} = d\omega$ for some 1-form ω. Expand ω as $\sum_i P_i\, dx_i$ and write $\omega = \omega_{\mathbf{P}}$ to reflect this expansion. Then direct calculation shows that $d\omega_{\mathbf{P}}$ is the 2-form $\theta_{\nabla\times\mathbf{P}}$ in the notation at the beginning of the proof. Therefore $\theta_{\mathbf{F}} = \theta_{\nabla\times\mathbf{P}}$, and by construction this means that $\mathbf{F} = \nabla \times \mathbf{P}$.■

Generalization to arbitrary smooth manifolds

In fact, one can state and prove de Rham's Theorem for every (second countable) smooth manifold if we use Conlon's approach to define differential forms (and related constructions) more generally; details are given in Chapters 6–8 of Conlon. The details of this generalization are beyond the scope of this course, so we shall only give a purely formal method for deriving the general case of de Rham's Theorem from the special case of open sets in $\mathbb{R}^n$ and a generalization of differential forms satisfying a few simple properties.

FACT 6. *The category of (second countable) smooth manifolds and smooth mappings has the following properties:*

(i) *It contains the category of open sets in $\mathbb{R}^n$ as a full subcategory.*

(ii) *The cochain complex functors $\wedge^*$ and S^*_{smooth} extend to this category, and likewise for the natural transformation $\theta^* : \wedge^* \to S^*_{\text{smooth}}$.*

(iii) *Every smooth manifold M^m is a smooth retract of some open set $U \subset \mathbb{R}^N$ for sufficiently large values of N.*

Property (i) follows directly from the construction of the category of smooth manifolds and smooth mappings, while (ii) clearly must hold in any reasonable extension of differential forms to smooth manifolds. Finally, (iii) is an immediate consequence of the Tubular Neighborhood Theorem for a smooth embedding of M in some $\mathbb{R}^N$; one reference is Lee, Proposition 10.20, page 256.

In view of the preceding discussion, the general case of de Rham's Theorem will be a consequence of the following very general result:

THEOREM 7. *Let $\mathcal{A}$ be a category, let $\mathcal{W} \subset \mathcal{A}$ be a full subcategory, and assume that every object in $\mathcal{A}$ is an $\mathcal{A}$-retract of an object in $\mathcal{W}$. Assume further that E and F are contravariant functors from $\mathcal{A}$ to the category of abelian groups and that $\theta : E \to F$ is a natural transformation. Then $\theta(X)$ is an isomorphism for all objects X in $\mathcal{A}$ if and only if it is an isomorphism for all objects X in $\mathcal{W}$.*

Proof. One implication is trivial, so we shall only look at the other case in which $\theta(X)$ is an isomorphism for all objects X in $\mathcal{W}$.

Suppose that X is an object of $\mathcal{A}$, choose a retract $i : X \to Y$, where Y is an object of $\mathcal{W}$, and let $r : Y \to x$ be such that $r \circ i = \mathrm{id}(X)$. Consider the following commutative diagram:

$$\begin{array}{ccccc} E(Y) & \xrightarrow{i^*} & E(X) & \xrightarrow{r^*} & E(Y) \\ \downarrow \theta_Y & & \downarrow \theta_X & & \downarrow \theta_X \\ F(Y) & \xrightarrow{i^*} & F(X) & \xrightarrow{r^*} & F(Y) \end{array}$$

Since $i^* \circ r^*$ is the identity on $E(X)$ and $F(X)$, it follows that i^* is onto and r^* is 1–1. To see that θ_X is 1–1, notice that $\theta_X(u) = \theta_X(v)$ implies $\theta_Y r^*(u) = r^*\theta_X(u) = r^*\theta_X(v) = \theta_Y r^*(v)$. Since θ_Y is an isomorphism it follows that $r^*(u) = r^*(v)$, which in turn implies $u = v$ because r^* is 1–1. To see that θ_X is onto, given $u \in F(X)$ use the surjectivity of i^* to write $u = i^*(v)$. Since θ_Y is an isomorphism it follows that $v = \theta_Y(w)$ for some w, and thus we have $u = i^*\theta_Y(v) = \theta_X i^*(w)$.■

V.5 : Multiplicative properties of de Rham cohomology

(Hatcher, §§ 3.1–3.2; Conlon, § D.3; Lee, Ch. 15)

DEFAULT HYPOTHESIS. Unless specifically stated otherwise, all cochain complexes, modules in this section are vector spaces over the real numbers, all algebraic morphisms are linear transformations, and all tensor products are taken over the real numbers.

As in the case of simplicial cup products, the Leibniz rule for for differential forms

$$d(\omega_1 \wedge \omega_2) \quad = \quad (d\omega_1) \wedge \omega_2 \ \pm \ \omega_1 \wedge d(\omega_2)$$

implies that the wedge of two closed forms is closed and the wedge of a closed form with an exact form is exact. Consequently there is a well defined (bilinear) cohomology wedge product

$$H^p_{\mathrm{DR}}(M) \otimes_{\mathbb{R}} H^q_{\mathrm{DR}}(M) \ \longrightarrow \ H^{p+q}_{\mathrm{DR}}(M)$$

sending $[\omega] \otimes [\theta]$ to $[\omega \wedge \theta]$ (where ω and θ are closed forms). It follows imeediately that this product makes the de Rham cohomology of a smooth manifold into a graded algebra and this structure is functorial. Since the de Rham and singular cohomology groups of a smooth manifold are isomorphic, it is natural to ask if the wedge product and cup product correspond under the isomorphism in de Rham's theorem, and it turns out that this is the case. We shall not give all the details of the argument; the references mentioned at appropriate points contain the omitted steps. Our approach will involve some explicit constructions involving simplicial chains and cochains which are taken from Eilenberg and Steenrod and also from the following classic text (which we shall simply call *Homology*):

> **S. MacLane.** *Homology* (Reprint of the first edition). Grundlehren der mathematischen Wissenschaften Bd. 114. Springer-Verlag, Berlin-New York, 1967.

LEMMA 1. *Let A be a p-simplex in $\mathbb{R}^n$ with vertices $a_0, \cdots, a_p$, and let B be a p-simplex in $\mathbb{R}^m$ with vertices $b_0, \cdots, b_q$. Then there is a simplicial decomposition of $A \times B \subset \mathbb{R}^n \times \mathbb{R}^m$ such that every point likes on at one $(p+q)$-simplex and an arbitrary $(p+q)$-simplex of the decomposition has vertices*

$$(a_{i_0}, b_{j_0}), \ \cdots \ , (a_{i_{p+q}}, b_{j_{p+q}})$$

where $i_t \geq i_{t+1}$, $j_t \geq j_{t+1}$ for all t and exactly one *of these two inequalities is* strict *for each t.*

For future reference, we note that the vertices of this decomposition for $A \times B$ have a standard lexicographic ordering obtained from the given orderings for the vertices of A and B.

Lemma 1 is a special case of the construction appearing in Section II.8 of Eilenberg and Steenrod.■

We shall also need an explicit singular (in fact, simplicial) chain

$$X(p,q) \quad \in \quad C_{p+q}(\Lambda_p \times \Lambda_q) \quad \subset \quad S_{p+q}(\Lambda_p \times \Lambda_q)$$

defined on page 243 of *Homology*. This chain contains plus or minus each of the affine ordered simplices mentioned in Lemma 1, and the sign is that of $\det T_\alpha$, where T_α is the unique affine map sending the vertices of Δ_{p+q} monotonically to those of the $(p+q)$-simplex $\alpha \subset \Lambda_p \times \Lambda_q$ (the assertion about signs requires a little work). The choice of signs leads to the following result:

LEMMA 2. *Let f be a smooth real valued function defined on an open neighborhood of $\Lambda_p \times \Lambda_q \subset \mathbb{R}^p \times \mathbb{R}^q$. Then*

$$\int_{\Lambda_p \times \Lambda_q} f(t)\, dt \quad = \quad \int_{X(p,q)} f(t)\, dt^1 \wedge \; \cdots \; \wedge dt^{p+q}$$

where the left hand side is the usual Riemann or Lebesgue integral and the right hand side is the differential forms integral.■

Again turning to page 743 of *Homology*, we see that there is a natural chain transformation

$$\gamma : S_*^{\mathrm{smooth}}(M) \otimes S_*^{\mathrm{smooth}}(N) \;\longrightarrow\; S_*^{\mathrm{smooth}}(M \times N)$$

such that if M and N are open neighborhoods of Λ_p and Λ_q in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively and $\sigma_k : \Lambda_k \to \mathbb{R}^k$ is the standard inclusion, then $\gamma(\sigma_p \otimes \sigma_q) = X(p,q)$; in fact, the map γ is an explicit chain inverse to the Alexander-Whitney map (see pages 743–744 of *Homology*).

NOTATIONAL CONVENTIONS. Given forms $\omega \in \wedge^p(M)$ and $\eta \in \wedge^q(N)$, the *external wedge* $\omega \times \eta \in \wedge^{p+q}(M \times n)$ is equal to

$$\left(p_M^{\#}\omega\right) \wedge \left(p_M^{\#}\omega\right) \quad \in \quad \wedge^{p+q}(M \times N) \; .$$

In coordinates, if $\omega = f(x)\, dx^1 \wedge \; \cdots \; \wedge dx^p$ and $\eta = g(y)\, dy^1 \wedge \; \cdots \; \wedge dy^q$, then

$$\omega \times \eta \quad = \quad f(x)\, g(y)\, dx^1 \wedge \; \cdots \; \wedge dx^p \wedge dy^1 \wedge \; \cdots \; \wedge dy^q \; .$$

Given a chain complex S_*, a commutative ring with unit $\mathbb{R}$, and cochains $f : S_p \to \mathbb{R}$, $g : S_q \to \mathbb{R}$, define the map $f \bowtie g : S_p \otimes S_q \to \mathbb{R}$ by the formula

$$f \bowtie g(u \otimes v) \quad = \quad f(u) \cdot g(v) \; .$$

We can now state and prove a key fact relating the cross product in singular cohomology and the external wedge product in de Rham cohomology.

PROPOSITION 3. *The following diagram is commutative:*

$$\begin{array}{ccc}
\wedge^p(M) \otimes \wedge^q(N) & \xrightarrow{\times} & \wedge^{p+q}(M \times N) \\
\big\downarrow \theta_M \otimes \theta_N & & \big\downarrow \theta_{M \times N} \\
S^p_{\mathrm{smooth}}(M) \otimes S^q_{\mathrm{smooth}}(N) & \longrightarrow & S^{p+q}_{\mathrm{smooth}}(M \times N) \\
\big\downarrow \bowtie & & \big\downarrow (\gamma | S_p \otimes S_q)^* \\
[S_p \otimes S_q]^* & \xrightarrow{=} & [S_p \otimes S_q]^*
\end{array}$$

In this diagram W^* denotes the dual space to the vector space W.

Proof. By the naturality properties of the constructions in the diagram, it suffices to consider the case in which M and N are open neighborhoods of the simplices $\Lambda_p,\ \Lambda_q \in \mathbb{R}^p, \mathbb{R}^q$ and to evaluate both composites applied to a tensor product of forms $\omega \otimes \eta$ on the universal example $\sigma_p \otimes \sigma_q$. Assume ω and η are given as in the notational conventions. Then the value of the composite $\bowtie \circ (\theta_M \otimes \theta_N) \circ (\omega \otimes \eta)$ at $\sigma_p \otimes \sigma_q$ is equal to

$$\int_{\sigma_p} \omega \cdot \int_{\sigma_q} \eta \;=\; \int_{\Lambda_p} f(x)\, dx \cdot \int_{\Lambda_q} g(y)\, dy \;=\; \int_{\Lambda_p \times \Lambda_q} f(x) \cdot g(y)\ dx\, dy$$

(the last equation follows from Fubini's Theorem). By Lemma 2, the last integral in the display is equal to

$$\int_{X(p,q)} f(x)\, g(y)\, dx^1 \wedge \cdots \wedge dy^q \;=\; \int_{X(p,q)} \omega \times \eta$$

and by definition the latter is equal to

$$(\gamma | S_p \otimes S_q)^* \circ \theta_{M \times N} \circ (\omega \otimes \eta) \quad \text{evaluated at} \quad \sigma_p \otimes \sigma_q$$

which is what we wanted to prove.■

The next result is nearly as important as the previous one for relating the cup and wedge products.

PROPOSITION 4. *Suppose that $r + t = p + q$ but $(s,t) \neq (p,q)$. Then $(\gamma | S_r \otimes S_t)^* \theta(\omega \otimes \eta) = 0$.*

Proof. Again by naturality it suffices to consider the case where M and N are open neighborhoods of the simplices $\Lambda_r,\ \Lambda_t \in \mathbb{R}^r, \mathbb{R}^t$ and to evaluate at $\sigma_r \otimes \sigma_t$. The hypothesis implies that either $r < p$ or $t < q$. Since $\dim M = r$ and $\dim N = t$, it follows that either $\wedge^p(M)$ or $\wedge^q(N)$ is trivial.■

The preceding results give us a cochain level formula relating the cross and external wedge products (and thus also for the cup and ordinary wedge products).

PROPOSITION 5. *In the setting above, let $\psi : S_*(M \times N) \to S_*(M) \otimes S_*(N)$ be the Alexander-Whitney map. Then $\theta_M(\omega) \times \theta_N(\eta) = \psi^* \circ \gamma^* \circ \theta_{M \times N}(\omega \times \eta)$.*

Proof. The left hand side is equal to $\psi^* \circ \rho(p,q) \circ \theta_M \otimes \theta_N(\omega \otimes \eta)$, where $\rho(p,q)$ projects $[S_*^{\text{smooth}}(M) \otimes S_*^{\text{smooth}}(N)]_{p+q}$ onto the direct summand $S_p^{\text{smooth}}(M) \otimes S_q^{\text{smooth}}(N)$. By Proposition 4 the composite $\psi^* \circ \gamma^* \circ \theta_{M \times N}(\omega \times \eta)$ is equal to $\rho(p,q)^* \circ (\gamma | S_p \otimes S_q)^* \circ \theta_{M \times N}(\omega \times \eta)$, and therefore $\theta_M(\omega) \times \theta_N(\eta) = \psi^* \circ \gamma^* \circ \theta_{M \times N}(\omega \times \eta)$ by Proposition 3.■

We can now state and prove the main result of this section:

THEOREM 6. *Let $\theta : H^*_{\text{DR}}(M) \to H^*(M)$ be the isomorphism in de Rham's Theorem, and let ω and η be closed forms on M. Then $\theta_M([\omega] \wedge [\eta]) = \theta_M([\omega]) \cup \theta_M([\eta])$.*

Proof. Let $M = N$ in the preceding discussion, and let $\Delta_M : M \to M \times M$ be the diagonal. Applying the cochain mapping $\Delta_M^{\#}$ to the right hand side of the equation in Proposition 5, we get $\theta_M(\omega) \cup \theta_M(\eta)$ on the cochain level. Applying $\Delta_M^{\#}$ to the left hand side, we get $\Delta_M^{\#} \circ \psi^* \circ \gamma^* \circ \theta_{M \times M}(\omega \times \eta)$. Since $\gamma \circ \psi$ is chain homotopic to the identity (γ is a a chain homotopy inverse to ψ), the conditions $d\omega = d\eta = 0$ imply that

$$\psi^* \circ \gamma^* \circ \theta_{M \times M}(\omega \times \eta) \;=\; \theta_{M \times M}(\omega \times \eta) \;+\; \delta z \qquad \text{for some} \quad z\ .$$

Therefore we have

$$\theta_M(\omega) \cup \theta_M(\eta) \;=\; \Delta_M^{\#}\left(\theta_{M \times M}(\omega \times \eta) \;+\; \delta z\right) \;=$$

$$\theta_M \circ \Delta_M^{\#}(\omega \times \eta) \ + \ \delta \Delta_M^{\#} z \quad = \quad \theta_M(\omega \wedge \eta) \ + \ \delta \Delta_M^{\#} z$$

where the last equation follows because $\omega \wedge \eta = \Delta_M^{\#}(\omega \times \eta)$. This means that if ω and η are closed forms, then the closed forms $\theta_M(\omega \wedge \eta)$ and $\theta_M(\omega) \cup \theta_M(\eta)$ determine the same singular cohomology class.■

V.6 : Path independence of line integrals

(Conlon, § 8.2; Lee, Chs. 11, 16)

In Section VIII.6 of `fundgp-notes.pdf` we proved that if U is an open subset of $\mathbb{R}^2$ and P and Q are functions with continuous partial derivatives in U satisfying

$$\frac{\partial P}{\partial y} \quad = \quad \frac{\partial Q}{\partial x}$$

at every point of U, then for each closed rectifiable curve Γ in U the line integral

$$\int_\Gamma P\,dx \ + \ Q\,dy$$

depend only on the (based or free) homotopy class of Γ, and that if Γ and Γ' are two piecewise smooth curves in U with the same endpoints $\mathbf{p}$ and $\mathbf{q}$ such that Γ and Γ' are homotopic by an endpoint preserving homotopy. Then

$$\int_\Gamma P\,dx + Q\,dy \quad = \quad \int_{\Gamma'} P\,dx + Q\,dy \ .$$

In fact, we showed that both of these results followed from the following result which is simultaneously a special case of both:

THEOREM 0. *Let U be a connected open subset of $\mathbb{R}^2$, and let P and Q be smooth functions on U with continuous partial derivatives which satsify the condition*

$$\frac{\partial P}{\partial y} \quad = \quad \frac{\partial Q}{\partial x}$$

at all points of U. If Γ is a piecewise smooth closed curve which starts and ends at $\mathbf{p}_0 \in U$ which is basepoint preservingly homotopic to a constant in U, then

$$\int_\Gamma P\,dx + Q\,dy \quad = \quad 0.$$

The goal of this section is to prove generalizations of the results from Section VIII.6 of the notes `fundgp-notes.pdf` to line integrals in open subsets of $\mathbb{R}^n$ for higher values of n; we shall do so by the combining the methods employed in the 2-dimensional case with the machinery of differential forms, the generalized Stokes' Theorem proved in this unit, and de Rham's Theorem.

As in the 2-dimensional case, it will be convenient to center the exposition around the following version of the main results:

THEOREM 1. *Let U be a connected open subset of $\mathbb{R}^n$, and let $P_1 \cdots, P_n$ be smooth functions on U with continuous partial derivatives which satsify the conditions*

$$\frac{\partial P_i}{\partial x_j} \quad = \quad \frac{\partial P_j}{\partial x_i}$$

at all points of U for all $i \neq j$. If Γ is a piecewise smooth closed curve which starts and ends at $\mathbf{p}_0 \in U$ which is basepoint preservingly homotopic to a constant in U, then

$$\int_\Gamma \sum_i P_i \, dx_i \quad = \quad 0.$$

Before proving this result, we shall state a few alternate versions and derive them from Theorem 1. The exposition is completely analogous to the corresponding material in the 205A notes.

THEOREM 2. *Let U and P_i be given as in Theorem* 1*, but suppose now that Γ and Γ' are two piecewise smooth curves in U with the same endpoints $\mathbf{p}$ and $\mathbf{q}$ such that Γ and Γ' are homotopic by an endpoint preserving homotopy. Then*

$$\int_\Gamma \sum_i P_i \, dx_i \quad = \quad \int_{\Gamma'} \sum_i P_i \, dx_i \quad .$$

Proof that Theorem 1 implies Theorem 2. In the setting of Theorem 2 the curve $\Gamma' + (-\Gamma)$ is a closed piecewise smooth curve that is homotopic to a constant because $\Gamma \simeq \Gamma'$ implies $[\Gamma' + (-\Gamma)] = [\Gamma + (-\Gamma)] = [\text{constant}]$. Therefore Theorem 1 implies that the line integral over this curve is zero. On the other hand, by the three properties of line integrals listed above, the line integral over $\Gamma' + (-\Gamma)$ is equal to the difference of the line integrals over Γ' and Γ. Combining these observations, we see that the line integrals over Γ' and Γ must be equal.■

The next result is often also found in multivariable calculus texts.

COROLLARY 3. *If in the setting preceding theorems we also know that the region U is* simply connected*, then the following hold:*

(i) *for every piecewise smooth closed curve Γ in U we have*

$$\int_\Gamma \sum_i P_i \, dx_i \quad = \quad 0 \; .$$

(ii) *for every pair of piecwise smooth curves Γ, Γ' with the same endpoints we have*

$$\int_\Gamma \sum_i P_i \, dx_i \quad = \quad \int_{\Gamma'} \sum_i P_i \, dx_i \quad .$$

The first part of the corollary follows from the triviality of the fundamental group of U, the conclusion of Theorem 1, and the triviality of line integrals over constant curve. The second part follows formally from the first in the same way that the Theorem 2 follows from Theorem 1.■

Finally, we have the following result concerning freely homotopic closed curves.

THEOREM 4. *Let U and P_i be given as in Theorems 1 and 2, but suppose now that Γ and Γ' are two piecewise smooth closed curves in U such that Γ and Γ' are* freely *homotopic. Then*

$$\int_{\Gamma} \sum_i P_i \, dx_i \quad = \quad \int_{\Gamma'} \sum_i P_i \, dx_i \quad .$$

In analogy with the 2-dimensional case, this proof will require some additional input, so the argument will be postponed until after the proof of Theorem 1 is completed.

Restatements using differential forms

The proof in the 2-dimensional case was based upon some consequences of Green's Theorem in the plane. In higher dimensions, it is more efficient to translate everything into differential forms and use the Generalized Stokes' Theorem of this unit instead. From this perspective, the integrands $\sum_i P_i \, dx_i$ correspond to differential 1-forms, and the line integral over a reasonable curve Γ can be written as the integral $\int_\Gamma \omega$, where $\omega = \sum_i P_i \, dx_i$. Once again, the Fundamental Theorem of Calculus and the Chain Rule imply that if Γ is a regular piecewise smooth curve defined on $[0,1]$ and f is a smooth function on U then

$$\int_{\Gamma} \sum_i \frac{\partial P_i}{\partial x_i} \, dx_i \quad = \quad f \circ \Gamma(1) \; - \; f \circ \Gamma(0)$$

and this is merely the 0-dimensional case of the Generalized Stokes' Formula.

Proof of Theorem 1. As in the preceding discussion, let ω denote the integrand of the line integrals under consideration. Then the conditions on the partial derivatives of the functions P_i translate into the vanishing of the exterior derivative $d\omega$.

Given (U, u_0) as in the statement of the theorem, let $\Theta(U, u_0)$ denote the set of all pairs (γ, Δ) where $\gamma : [0,1] \to U$ is a regular piecewise smooth curves such that $\gamma(0) = \gamma(1) = u_0$ and

$$\Delta \quad = \quad \{ 0 \; = \; t_0 \; < \; \cdots \; < \; t_m \; = \; 1 \}$$

defines a partition of $[0,1]$ into subintervals such that the restriction of γ to each subinterval $[t_{i-1}, t_i]$ is a regular smooth curve. For each i let T_i be the regular smooth curve defined by composing $\gamma|[t_{i-1}, t_i]$ with the standard endpoint and order preserving linear map $[0,1] \to [t_{i-1}, t_i]$, and define a mapping

$$\mathrm{Chain}_{(U,u_0)} : \Theta(U, u_0) \; \longrightarrow S_1^{\mathrm{smooth}}(U)$$

sending (γ, Δ) to $\sum_i T_i$. Since γ is a closed curve the boundary of the chain $\sum_i T_i$ is zero, and therefore the image of $\mathrm{Chain}_{(U,u_0)}$ lies in the subgroup $Z_1^{\mathrm{smooth}}(U)$ of cycles in $S_1^{\mathrm{smooth}}(U)$. Therefore $\mathrm{Chain}_{(U,u_0)}$ yields a mapping $\eta_{(U,u_0)}$ from $\Theta(U, u_0)$ to $H_1^{\mathrm{smooth}}(U)$.

We can now define a concatenation operation on $\Theta(U, u_0)$ which sends the ordered pair $\big((\gamma, \Delta), \; (\gamma', \Delta') \big)$ to (γ', Δ') where $\gamma + \gamma'$ is the concatenated curve and $\Delta + \Delta'$ is the corresponding partition of $[0,1]$ given by shrinking Δ and Δ' to $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ respectively, and it follows immediately that the previously constructed mapping $\eta_{(U,u_0)}$ sends $(\gamma, \Delta) + (\gamma', \Delta')$ to $\eta_{(U,u_0)}(\gamma, \Delta) + \eta_{(U,u_0)}(\gamma', \Delta')$.

Consider now the following commutative diagram, in which the vertical map at the left sends (γ, Δ) to the homotopy class $[\gamma]$ and the map h is the Hurewicz homomorphism:

$$\begin{array}{ccc} \Theta(U, u_0) & \xrightarrow{\eta} & H_1^{\mathrm{smooth}}(U)) \\ \downarrow & & \downarrow \\ \pi_1(U, u_0) & \xrightarrow{h} & H_1(U)) \end{array}$$

We can now complete the proof as follows: The basic properties of line integrals imply that the line integral of ω along γ is equal to the integral of γ with respect to the chain $\mathrm{Chain}(\gamma, \Delta) = \sum_i T_i$. Since $d\omega = 0$, the Generalized Stokes' Theorem implies that the integral $\int_{\mathrm{Chain}(\gamma,\Delta)} \omega$ only depends on the image of (γ, Δ) in $H_1^{\mathrm{smooth}}(U)$. If γ is basepoint preserving homotopic to the constant map whose value everywhere is u_0, then the class of γ in $\pi_1(U, u_0)$ is trivial and hence we can use the diagram to conclude that $h \circ [\gamma] = 0$ and hence $\eta(\gamma, \Delta) = 0$, and therefore by the preceding sentence we know that $\int_\gamma \omega = 0$, which is what we wanted to prove.■

Proof that Theorem 1 implies Theorem 4. We do not know whether or not the freely homotopic closed curves γ_0 and γ_1 start and end at the same point, so assume that γ_i starts and ends at u_i for $i = 0, 1$. Choose appropriate partitions Δ_i such that γ_i is smooth on each subinterval determined by Δ_i for $i = 0, 1$. Since γ_0 and γ_1 are freely homotopic, the commutative diagram implies that

$$\eta_{(U,u_0)}(\gamma_0, \Delta_0) \quad = \quad \eta_{(U,u_1)}(\gamma_1, \Delta_1) \qquad \text{in} \qquad H_1^{\mathrm{smooth}}(U)\ .$$

As in the proof of Theorem 1, the integrand $\sum_i P_i\, dx_i$ corresponds to a closed 1-form ω, and therefore in this case the Generalized Stokes' Theorem implies that the integrals of ω over the chains $\int_{\mathrm{Chain}(\gamma_0,\Delta_0)} \omega$ and $\int_{\mathrm{Chain}(\gamma_1,\Delta_1)} \omega$ are equal As in the proof of Theorem 1, we know that these inegrals are respectively equal to the line integrals $\int_{\gamma_0} \omega$ and $\int_{\gamma_1} \omega$, and therefore these two line integrals must also be equal.■

Some classical implications

Frequently one sees the 3-dimensional case of the following result in multivariable calculus texts:

THEOREM 5. *Let $n \geq 3$, and suppose that U is obtained from $\mathbb{R}^n$ by removing finitely many points. If $\mathbf{F} = (P_1, \cdots, P_n)$ is a smooth vector field on U such that*

$$\frac{\partial P_i}{\partial x_j} \quad = \quad \frac{\partial P_j}{\partial x_i}$$

for all $i \neq j$, then there is a smooth function g on U such that $\nabla g = \mathbf{F}$. In particular, if Γ is a regular piecewise smooth curve in U, then the value of the line integral

$$\int_\Gamma \sum_i P_i\, dx_i$$

depends only upon the endpoints of the curve Γ.

In particular, a result of this type is formulated as Theorem 7 on page 551 of Marsden and Tromba.

Theorem 5 contrasts sharply with the case $n = 2$, and the easiest way to explain the difference is to note that the complement of a finite subset in $\mathbb{R}^n$ is simply connected if $n = 3$ but is not simply connected if $n = 2$. We shall give a simpler (but less elementary) argument which only requires us to know that $H^1(U; \mathbb{R})$ is trivial if $n \geq 3$. By the Universal Coefficient Theorem relating integral homology to real cohomology, we only need to prove the following:

LEMMA 6. *Let $n \geq 3$, and suppose that U is obtained from $\mathbb{R}^n$ by removing a set X which contains exactly k points. Then the singular homology groups of $U = \mathbb{R}^n - X$ are given by $H_j(\mathbb{R}^n - X) \cong \mathbb{Z}$ if $k = 0$, $H_j(\mathbb{R}^n - X) \cong \mathbb{Z}^k$ if $k = n - 1$, and $H_j(\mathbb{R}^n - X) \cong 0$ otherwise.*

Proof that Lemma 6 implies Theorem 5. By Lemma 10 we know that $H_1(U = \mathbb{R}^n - X)$ is trivial because $n \geq 3$, and by the Universal Coefficient Theorem we know that $H^1(U; \mathbb{R}) \cong \mathrm{Hom}\,(H_1(U), \mathbb{R})$; therefore $H^1(U; \mathbb{R})$ is trivial. Since $H^*_{\mathrm{DR}}(U) \cong H^*(U; \mathbb{R})$ by de Rham's Theorem, it follows that $H^1_{\mathrm{DR}}(U)$ is trivial and therefore every closed 1-form over U is exact; *i.e.*, if $d\omega = 0$ then $\omega = dg$ for some g.

Let ω be the 1-form $\sum_i P_i\, dx_i$; the hypothesis on the functions P_i is equivalent to the identity $d\omega = 0$, and therefore if this identity holds we can apply the preceding paragraph to conclude that $\omega = dg$ for some smooth function g. If we translate this back into the language of vector fields, we see that the original vector field $\mathbf{F}$ is equal to ∇g, proving the first assertion in the conclusion of the theorem. The second assertion now follows because the line integral in question has the form $\int_\Gamma \nabla g \cdot d\mathbf{x}$ and we have already noted that the values of such line integrals only depend upon the endpoints of Γ.∎

Proof of Lemma 6. For each $x \in X$ let V_x be the open neighborhood of radius r centered at x; choose r to be smaller than half the minimum distance between points of X (the minimum exists by the finiteness of X, and let $V = \cup_x U_x$, so that $\mathbb{R}^n = V \cup (\mathbb{R}^n - X))$ and $V \cap X = \cup_x V_x - \{x\}$. Then by excision, the splitting of the homology of X into the homology of its arc componenents, and Theorem VII.1.7 in `algtop-notes.pdf` we know that

$$H_j(\mathbb{R}^n, \mathbb{R}^n - X) \cong H_j(U, U - X) \cong H_j\left(\cup_x V_x, \cup_x V_x - \{x\}\right) \cong$$

$$\bigoplus_x H_j\left(V_x, V_x - \{x\}\right) \cong \mathbb{Z}^k \quad \text{or} \quad 0$$

where the group is zero unless $j = n$, in which case it is isomorphic to $\mathbb{Z}^k$. We can now recover the homology groups $\mathbb{R}^n - X$ from the long exact homology sequence for $(\mathbb{R}^n, \mathbb{R}^n - X)$ and the fact that $H_j(\mathbb{R}^n)$ is $\mathbb{Z}$ if $j = 0$ and zero otherwise.∎

Similar conclusions hold if U is obtained from $\mathbb{R}^n$ (where $n \geq 3$) by deleting an infinite sequence of isolated points $\{\mathbf{p}_1, \mathbf{p}_2, \cdots\}$. The main difference in the argument is that the open disk V_k centered at $\mathbf{p}_k$ must have a radius r_k such that for each $j \neq k$ we have $|\mathbf{p}_j - \mathbf{p}_k| > r_k$; we can always find such positive radii if we have a sequence of isolated points.∎