

Cellular Homology Diagrams

(1)

$$\underline{X_n = X}$$

$$\begin{array}{ccccccc}
 & & & & & 0 = H_{n-1}(X_{n-2}) & \\
 & & & & & \downarrow & \\
 0 = H_n(X_{n-1}) & \rightarrow & H_n(X_n) & \xrightarrow{j[n]} & H_n(X_n, X_{n-1}) & \xrightarrow{\partial[n]} & H_{n-1}(X_{n-1}) \\
 & & & & & & \downarrow j[n-1] \\
 & & & & & & 0 = H_{n-1}(X_{n-2})
 \end{array}$$

The vertical and horizontal lines are exact

$$\text{So } H_n(X) \cong \text{Ker } \partial[n] =$$

Also

$$H_n(X) \cong \text{Image } j[n]$$

$$\text{Ker } d_n = j[n-1] \circ \partial[n]$$

$$\text{Since } j[n-1] \text{ is } 1-1$$

$$\text{Hence } H_n(X) \cong H_n(C_*(X, \mathbb{Z}), d_*)$$

(see advanced notes 2012.pdf,
p. 28.)

(2)

$$0 = H_{m-1}(X_{m-2})$$



$$H_m(X_m, X_{m-1}) \xrightarrow{\partial[n]} H_{m-1}(X_{m-1}) \xrightarrow{\text{onto}} H_{m-1}(X_m) \rightarrow 0 = H_{m-1}(X_m, X_{m-1})$$

The vertical
and
horizontal lines

$$\begin{array}{c} 1-1 \downarrow j[n-1]_* \\ H_{m-1}(X_{m-1}, X_{m-2}) \end{array}$$

are exact.

$$\text{Hence } H_{m-1}(X_m) \cong H_{m-1}(X_{m-1}) / \text{Image } \partial[n]$$

$$\cong \text{Image } j[n-1]_* / \text{Image } d_m. \text{ But}$$

the results of the first page (or induction!) imply that

$$\text{Image } j[n-1]_* = \text{Kernel } d_{m-1}, \text{ so}$$

$$H_{m-1}(X_m) \cong \text{Kernel } d_{m-1} / \text{Image } d_m =$$

$$H_{m-1}(C_*(X, \mathcal{E}), d_*) , \text{ Q.E.D.}$$