De Rham cohomology for $S^{n-1} \times \mathbb{R} \cong \mathbb{R}^n - \{0\}$

By de Rham's Theorem we know that

$$H^{n-1}_{DR}(S^{n-1} \times \mathbb{R}) \cong H^{n-1}(S^{n-1} \times \mathbb{R}; \mathbb{R}) \cong \mathbb{R},$$

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so there is a closed $(n-1)$-form on $S^{n-1} \times \mathbb{R}$ which is not exact. Here is an explicit example; we identify $S^{n-1} \times \mathbb{R}$ with $\mathbb{R}^n - \{0\}$ as usual.

$$\omega = \sum_{k=1}^{n} \frac{(-1)^{k+1} \cdot x^k}{\left(\sum (x^i)^2\right)^{n/2}} \, dx^1 \wedge \ldots \underbrace{\widehat{dx^k}}_{\text{OMIT}} \ldots \wedge dx^n .$$

Verifying that $d\omega = 0$ is left as an exercise. To see ω is not exact, consider its integral over $S^{n-1} \cong \partial \Lambda_n$. As in the case $n = 3$, the integral $\int_{S^{n-1}} \omega$ is the "hypervolume" of S^{n-1}, which is positive, but Stokes' Formula implies that if $\omega = d\theta$, then $\int_{S^{n-1}} \omega = \int_{S^{n-1}} d\theta = \int_{\partial S^{n-1}} \theta = \int_{\emptyset} \theta$

which is zero.