page 92, lines 21-22

Proof that $\delta(f \cup g) = (\delta f) \cup g + (-1)^p f \cup (\delta g)$ where $f : S_p(X) \longrightarrow \mathbb{D}$, $g : S_q(X) \longrightarrow \mathbb{D}$.

Idea Prove they have the same value at a free generator $T : \Delta_{p+q+1} \longrightarrow X$ of $S_{p+q+1}(X)$.

$\boxed{\text{Given } i_0 < \cdots < i_r, \text{ let } v_{i_0} \cdots v_{i_r} = \text{restriction of } T \text{ to simplex w/ vertices } e_{i_j}}$

LHS $\delta(f \cup g)(T) =$

$$\sum_i (-1)^i f \cup g (v_0 \cdots \widehat{v_i} \cdots v_{p+q+1}) =$$

$$\sum_{i \le p} (-1)^i f(v_0 \cdots \widehat{v_i} \cdots v_{p+1})\, g(v_{p+1} \cdots v_{p+q+1}) +$$

$$\sum_{i \ge p+1} (-1)^i f(v_0 \cdots v_p)\, g(v_p \cdots \widehat{v_i} \cdots v_{p+q+1})$$

RHS Do pieces separately:

$(\delta f) \cup g (T) =$

$$\sum_{i=0}^{p+1} (-1)^i f(v_0 \cdots \widehat{v_i} \cdots v_{p+1})\, g(v_{p+1} \cdots v_{p+q+1})$$

page 92, lines 21-22 continued

$$(-1)^p f \cup \delta g\,(T) =$$

$$(-1)^p \sum_{i=p}^{p+q+1} (-1)^{i-p} f(v_0 \dots v_p)\, g(v_p \dots \hat{v}_i \dots v_{p+q+1})$$

If we subtract $\delta(f \cup g)(T)$ from $\delta f \cup g\,(T) + (-1)^p f \cup \delta g\,(T)$ we are left with the $p+1$ term in the sum for $\delta f \cup g\,(T)$ and the p term in the sum for $(-1)^p f \cup \delta g\,(T)$, which is

$$(-1)^{p+1} f(v_0 \dots v_p)\, g(v_{p+1} \dots v_{p+q+1}) +$$
$$(-1)^p f(v_0 \dots v_p)\, g(v_{p+1} \dots v_{p+q+1})$$

and since $(-1)^{p+1} + (-1)^p = 0$ these cancel each other. Hence the two expressions have the same value at T, which was what we wanted to prove.