page 98, line −5 to page 99, line 2

The formula discussed on these lines is

$$\langle \delta\alpha, y \rangle = \langle \alpha, \partial y \rangle$$

where all coefficients are in the field $\mathbb{F}$, with

$$\alpha \in H^{p-1}(A) \qquad \delta : H^{p-1}(A) \longrightarrow H^{p}(X,A)$$

$$y \in H_p(X,A) \qquad \partial : H_p(X,A) \longrightarrow H_{p-1}(A)$$

and $\langle\ ,\ \rangle$ = Kronecker index.

To verify this equation, use the definitions in terms of representative cycles and cocycles. Represent y by $c \in S_p(X)$ such that $dc \in S_{p-1}(A)$, and α by $g \in S^{p-1}(X)$ such that $\delta g = g \circ d \,|\, S_p(A) = 0$. (Note that $S^{p-1}(X) \xrightarrow{\text{onto}} S^{p-1}(A)$, so every cocycle on A comes from a cochain on X). By definition, $\langle \alpha, \partial y \rangle$ is then equal to $g(dc) = g \circ d(c) = \delta g(c)$, which is equal to $\langle \delta\alpha, y \rangle$.

page 99, lines -12 to -11

Verification that $d^{A\otimes B}_{n-1} \circ d^{A\otimes B}_{n} = 0$.

Details: Since $(A\otimes B)_n$ is generated by classes $y\otimes z$ where $y \in A_p$ and $z \in B_{q=n-p}$ for some p, it suffices to check that the composite sends these elements to zero.

We have $d(y\otimes z) = dy \otimes z + (-1)^p y \otimes dz$, so $dd(y\otimes z) = d(dy\otimes z + (-1)^p y \otimes dz) =$
$ddy \otimes z + (-1)^{p-1} dy \otimes dz + (-1)^p dy \otimes dz +$
$(-1)^{p-1}(-1)^p y \otimes ddz$. The first and last terms are zero because $dd = 0$, and the middle two terms cancel each other because $(-1)^p = -(-1)^{p-1}$.
Thus $dd = 0$ in $A\otimes B$, which is what we wanted to verify.

page 99, line -9

Proof of Proposition 1.

(i) Suppose $0 \longrightarrow K_* \longrightarrow A_* \longrightarrow C_* \longrightarrow 0$ is short exact and B_* is free in each dim. [As earlier on p. 99, assume all groups are zero in negative dims]. Note that if F is a free module and $0 \longrightarrow K \longrightarrow A \longrightarrow C \longrightarrow 0$ is short exact, then $F \cong \bigoplus_{\gamma \in \Gamma} \mathbb{D}_\gamma \Rightarrow$

$$0 \longrightarrow K \otimes_{\mathbb{D}} F \longrightarrow A \otimes_{\mathbb{D}} F \longrightarrow C \otimes_{\mathbb{D}} F \longrightarrow 0$$

is just a direct sum of the form

$$0 \longrightarrow \bigoplus_\gamma K_\gamma \longrightarrow \bigoplus_\gamma A_\gamma \longrightarrow \bigoplus_\gamma C_\gamma \longrightarrow 0$$

and is short exact (more generally, a direct sum of short exact sequences is short exact). Thus for each $n \geq 0$ the n-dimensional modules in the tensored chain complexes are given by the short exact sequence page 99, line -9, continued

$$0 \longrightarrow \bigoplus_{p\geq 0} K_p \otimes B_{n-p} \longrightarrow \bigoplus_{p\geq 0} A_p \otimes B_{n-p} \longrightarrow \bigoplus_{p\geq 0} C_p \otimes B_{n-p} \longrightarrow 0$$

and hence $0 \to K_* \otimes B_* \to \underset{A_*\otimes B_*}{K_* \otimes A_*} \to \underset{C_*\otimes B_*}{K_* \otimes C_*} \to 0$

is short exact.

(ii) Now assume K_*, A_*, C_* are free in each dimension. Then for all p we have $A_p \cong C_p \oplus K_p$ and the n-dim modules in

$$0 \to K_* \otimes B_* \longrightarrow A_* \otimes B_* \longrightarrow C_* \otimes B_* \to 0$$

are given by the short exact sequence

$$0 \longrightarrow \bigoplus_{p\geq 0} K_p \otimes B_{n-p} \longrightarrow \bigoplus_{p\geq 0} (K_p \oplus C_p) \otimes B_{n-p} \longrightarrow \bigoplus_{p\geq 0} C_p \otimes B_{n-p} \to 0$$

which is in fact a <u>split</u> short exact sequence.

Although the chain complex sequence

$$0 \to K_* \otimes B_* \longrightarrow A_* \otimes B_* \longrightarrow C_* \otimes B_* \longrightarrow 0$$

might not be (and usually is not) split, we can still conclude that it is short exact in the chain complex category.

page 100, lines 1-9

Details for the proof of Proposition 2.

(i) If $x \in A_p$ and $y \in B_q$ are cycles, then $x \otimes y$ is also a cycle.

Proof. $dx = 0$ and $dy = 0 \Rightarrow$

$$d(x \otimes y) = \underbrace{dx}_{0} \otimes y + (-1)^p x \otimes \underbrace{dy}_{0} = 0 + 0 = 0.$$

(ii) If $x, x' \in A_p$ and $y, y' \in B_q$ are cycles such that $x - x' = dw$ and $y - y' = dz$ for some w and z, then $[x \otimes y] = [x' \otimes y']$ in $H_{p+q}(A \otimes B)$.

Proof. $d(w \otimes y) = (x - x') \otimes y + (-1)^{p+1} w \otimes \underbrace{dy}_{0} = x \otimes y - x' \otimes y$, so $[x \otimes y] = [x' \otimes y]$ in $H_*(A \otimes B)$. Likewise, $d(x' \otimes z) = \underbrace{dx'}_{0} \otimes z + (-1)^p x' \otimes (y - y') = (-1)^p (x' \otimes y - x' \otimes y')$, so $[x' \otimes y] = [x' \otimes y']$ in $H_*(A \otimes B)$.

page 100, lines 1-9, continued

The preceding shows that the map

$$[x]\otimes[y] \xrightarrow[\text{“}\otimes\text{”}]{\mu_*} [x\otimes y] \text{ is well-defined.}$$

(iii) The map $\uparrow$ is bilinear.

This is easy given the well-definition of μ_*: $([x_1]+[x_2])\otimes[y] = [x_1+x_2]\otimes[y] = [(x_1+x_2)\otimes y] = [(x_1\otimes y)+(x_2\otimes y)] = ([x_1]\otimes[y]) + ([x_2]\otimes[y])$, and similarly we have $[x]\otimes([y_1]+[y_2]) = ([x]\otimes[y_1]) + ([x]\otimes[y_2])$.

Furthermore, if $c\in\mathbb{D}$ then $c\cdot([x]\otimes[y]) = c[x\otimes y] = [c(x\otimes y)] = [(cx)\otimes y] = [cx]\otimes[y]$, and similarly we have $c([x]\otimes[y]) = [x]\otimes[cy]$ (recall that $c(a\otimes b) = (ca)\otimes b = a\otimes cb$ in the tensor product $M\otimes_{\mathbb{D}}N$, where $\mathbb{D}$ is a commutative ring with unit).