page 124, lines -15 to -14

Verification that the permutation $\tau_{p,q}$

$$\begin{array}{c} 1 \ldots p \,\vdots\, p+1 \ldots p+q \\ \downarrow \\ p+1 \ldots p+q \,\vdots\, 1 \ldots p \end{array}$$

has sign $(-1)^{pq}$.

Follow the suggestion on line -14, fixing p and proceeding by induction on q.

$q=1$ The permutation is just the cycle $(1 \ldots p \; p+1)$, which has sign $(-1)^p$.

Assume for $q=k$, and suppose $q=k+1$.

Then $\operatorname{sgn} \tau_{p,k} = (-1)^{pk}$ and $\tau_{p,k}$ is

$$\begin{array}{c} 1 \ldots p \,\vdots\, p+1 \ldots p+k \,\vdots\, p+k+1 \\ \downarrow \\ p+1 \ldots p+k \,\vdots\, 1 \ldots p \,\vdots\, p+k+1 \end{array}$$

The latter implies $\tau_{p,k+1} = (k+1 \ldots p+k+1) \circ \tau_{p,k}$ and hence has sign $= (-1)^p \cdot (-1)^{pk} = (-1)^{p(k+1)}$ completing the verification of the inductive step.

page 124, lines −11 to −10

Verification of Proposition 4.

By construction there is a commutative diagram (Ω = quotient projection)

$$\begin{array}{ccc} \mathrm{Cov}_p(U) \times \mathrm{Cov}_q(U) & \xrightarrow{\otimes} & \mathrm{Cov}_{p+q}(U) \\ \downarrow \Omega\times\Omega & & \downarrow \Omega \\ \Lambda^p(U) \times \Lambda^q(U) & \xrightarrow{\wedge} & \Lambda^{p+q}(U) \end{array}$$

for all p and q, so if $\overline{\lambda}, \overline{\omega}, \overline{\theta} \in \mathrm{Cov}_*(U)$ project to λ, ω, θ we have

$$(\theta \wedge \omega) \wedge \lambda = (\Omega\overline{\theta} \wedge \Omega\overline{\omega}) \wedge \Omega\overline{\lambda} =$$

$$\Omega(\overline{\theta} \otimes \overline{\omega}) \wedge \Omega(\overline{\lambda}) = \Omega((\overline{\theta} \otimes \overline{\omega}) \otimes \overline{\lambda}) \quad \frac{\text{since } \otimes \text{ is}}{\text{associative}}$$

$$\Omega(\overline{\theta} \otimes (\overline{\omega} \otimes \overline{\lambda})) = \Omega(\overline{\theta}) \wedge \Omega(\overline{\omega} \otimes \overline{\lambda}) =$$

$$\Omega(\overline{\theta}) \wedge (\Omega\overline{\omega} \wedge \Omega\overline{\lambda}) = \theta \wedge (\omega \wedge \lambda).$$

page 125, lines 1-6

Proof of Theorem 5

(i) Since both the left and right hand sides are $\mathbb{R}$-bilinear with respect to θ and ω, it suffices to verify this result for sets of forms θ_α and ω_α which span $\Lambda^p(U)$ and $\Lambda^q(U)$. The obvious choices are forms given by $f\,dx^{i_1}\wedge\ldots\wedge dx^{i_p}$ and $g\,dx^{j_1}\wedge\ldots\wedge dx^{j_q}$ where $f, g \in C^\infty(U)$.

$\boxed{\text{PRELIMINARY Note}}$

The relationship

$$\begin{matrix}\text{exterior derivative} \\ \text{of coord. fcn} \\ x^i\end{matrix} = \text{"}dx^i\text{"}$$

looks tautological, but it isn't quite. One has $d(x^i) = \sum \frac{\partial x^i}{\partial x^j}$ "dx^j" which reduces to "dx^i" because $\frac{\partial x^i}{\partial x^j} = \delta_{ij}$ (Kronecker delta).

<u>page 125, lines 1-6 continued</u>

So we need to compare

$$d((f\,dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge(g\,dx^{j_1}\wedge\cdots\wedge dx^{j_q})$$ and

$$d(f\,dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge(g\,dx^{j_1}\wedge\cdots\wedge dx^{j_q})+$$

$$(-1)^p(f\,dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge d(g\,dx^{j_1}\wedge\cdots\wedge dx^{j_q}).$$

The first of these is

$$d(fg)\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}\wedge dx^{j_1}\cdots dx^{j_q}=$$

$$\sum f\frac{\partial g}{\partial x^k}dx^k\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q}+$$

$$\sum\frac{\partial f}{\partial x^l}g\,dx^l\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q}=$$

$$(-1)^p(f\,dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge(dg\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q})+$$

$$(df\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge(g\,dx^{j_1}\wedge\cdots\wedge dx^{j_q})=$$

$$(-1)^p\,\theta\wedge d\omega+d\theta\wedge\omega.$$

page 125, lines 1-6 continued

(ii) Since $d \circ d$ is $\mathbb{R}$-linear, as in (i) it suffices to prove it for a form of type $f\, dx^{i_1} \wedge \dots \wedge dx^{i_p}$.

It is routine to check that $d\,dx^i = 0$ for all i, and hence by (i) we have $d(dx^{i_1} \wedge \dots \wedge dx^{i_p}) = 0$.

Now $df = \sum_k \frac{\partial f}{\partial x^k} dx^k \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}$ and

$$ddf = \sum_{j,k} \frac{\partial^2 f}{\partial x^j \partial x^k} dx^j \wedge dx^k \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} =$$

$$\sum_{j<k} + \sum_{j=k} + \sum_{j>k}.$$

The middle term vanishes because $dx^j \wedge dx^j = 0$, and the first and third cancel because

$$\frac{\partial^2 f}{\partial x^j \partial x^k} dx^j \wedge dx^k = -\frac{\partial^2 f}{\partial x^k \partial x^j} dx^k \wedge dx^j \quad \text{if } j \neq k.$$

page 126, lines 10-12

(i) In this case we have an isomorphism of free $C^\infty(U)$ modules

$$\Phi_1 : \text{Vector Fields} \to 1\text{-forms}$$

with Φ_1 of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ equal to dx, dy, dz.

Then $\nabla f = \frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial}{\partial z}$, and

$$\Phi_1(\nabla f) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

(ii) In this case Φ_2 sends $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ to $dy \wedge dz$, $dz \wedge dx$ [NOTE!!], $dx \wedge dy$ and

$$d\,\Phi_1\left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}\right) = d(P\,dx + Q\,dy + R\,dz)$$

$$= \frac{\partial P}{\partial y}dy\wedge dx + \frac{\partial P}{\partial z}dz\wedge dx + \frac{\partial Q}{\partial x}dx\wedge dy + \frac{\partial Q}{\partial z}dz\wedge dy + \frac{\partial R}{\partial x}dx\wedge dz + \frac{\partial R}{\partial y}dy\wedge dz = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dy\wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dz\wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\wedge dy$$

page 126, lines 10-12 continued

and the latter is equal to

$$\Phi_2\Big(\underbrace{\nabla\times}_{\text{CURL}}\Big(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}\Big)\Big)$$

(iii) $d\,\Phi_2\left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}\right) =$

$d\left(P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy\right) =$

$\frac{\partial P}{\partial x}\,dx\wedge dy\wedge dz + \frac{\partial Q}{\partial y}\,dy\wedge dz\wedge dx + \frac{\partial R}{\partial z}\,dz\wedge dx\wedge dy$

$= \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\wedge dy\wedge dz =$

$$\Phi_3\Big(\underbrace{\nabla\cdot}_{\text{DIV}}\Big(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}\Big)\Big).$$

page 139, lines −12 to −5

Direct products of cochain complexes

Let $\{(C_\alpha, \delta_\alpha)\}$ be an indexed family of cochain complexes. Then $(\prod C_\alpha, \prod \delta_\alpha)$ is a chain complex because $(\prod \delta_\alpha)^2 \underset{\text{of direct product}}{\overset{\text{functoriality}}{=}}$

$$\prod \delta_\alpha^2 = \prod 0 = 0.$$

Claim The projection maps

$\prod_\alpha H^*(C_\alpha) \to H^*(C_\gamma)$ some γ induce an isomorphism

$$H^*\left(\prod_\alpha C_\alpha\right) \to \prod_\alpha H^*(C_\alpha).$$

One needs to check that $(x_\alpha) \in \prod C_\alpha$ is a cocycle/coboundary $\Longleftrightarrow$ each x_γ is,

and then one can use the fact that if A_α is a submodule of B_α for all α, then

$$\prod_\alpha (A_\alpha / B_\alpha) \cong \left[\prod_\alpha A_\alpha\right] / \left[\prod_\alpha B_\alpha\right].$$