# Dual spaces, annihilators and subquotients

In the proof of Proposition IV.2.4 on page 98, on lines 13–15 we use a relationship involving dual spaces and subquotients of a vector space. In this document we shall state and prove this relationship.

THE BASIC SETTING. Let $V$ by a vector space over the field $\mathbb{F}$, suppose we are given vector subspaces $U_2 \subset U_1 \subset v$, and let $\alpha : V \to V/U_2$ denote the quotient projection. As in the proof of Propsition IV.2.4, if $W$ is a vector subspace of $V$ then $W^\dagger$ will denote the annihilator of $W$, which is a vector subspace of the dual space $V^*$.

Define a linear transformation

$$\varphi : U_2^\dagger / U_1^\dagger \quad \longrightarrow \quad (U_1/U_2)^*$$

as follows: Given $f \in U_2^\dagger$, let $f' : V/U_2 \to \mathbb{F}$ be the unique linear functional such that $f' \circ \alpha = f$, and set $\varphi(f)$ equal to $f'|(U_1/U_2)$.

**CLAIM.** *The map $\varphi$ is a vector space isomorphism.*

**Proof.** We shall first prove that $\varphi$ is 1–1. Suppose that $\varphi(f) = 0$; then the restriction of $f'$ to $U_1/U_2$ is zero, and therefore the restriction of $f$ to $U_1$ is zero. But this means that $f \in U_1^\dagger$, so that $\varphi(f) = 0$ on $(U_1/U_2)^*$.

Next, we shall prove that $\varphi$ is onto. Given a linear functional $g : U_1/U_2 \to \mathbb{F}$, one can always find an extension of $g$ to $G : V/U_2 \to \mathbb{F}$, and the composite $G \circ \alpha \in V^*$ clearly belongs to $U_2^\dagger$. By construction, $\varphi(G \circ \alpha) = g$.■