The Euler characteristic formula

The objective is to give a relatively brief and self-contained proof of the following result:

THEOREM. *Let C_* be a chain complex such that each C_j is finitely generated and free abelian, and assume that $C_j = 0$ if $j < 0$ or $j > n$. Furthermore, assume that each $H_k(C)$ is free abelian (it must be finitely generated by standard results in algebra). Then the ranks of the free abelian groups satisfy the formula*

$$\sum_k (-1)^k \operatorname{rank}(C_k) \quad = \quad \sum_k (-1)^k \operatorname{rank}\,(H_k(C))\ .$$

The Euler Formula for polyhedra is an immediate consequence of this.

Proof. We shall repeatedly use the following fact, which is proved in first year graduate algebra courses:

$(*)$ *If* $0 \to A \to B \to C \to 0$ *is a short exact sequence of finitely generated free abelian groups, then* $\operatorname{rank}(B) = \operatorname{rank}(C) + \operatorname{rank}(A)$.

The following notational conventions will simplify our computations:

(i) Set c_k equal to the rank of C_k.

(ii) Set b_k equal to the rank of d_k.

(iii) Set z_k equal to the rank of the kernel of d_k.

(iv) Set h_k equal to the rank of $H_k(C)$.

It follows immediately that these numbers are defined for all q and are equal to zero if $k < 0$ or $k > n$. The theorem can then be restated in the form

$$\sum_k (-1)^k c_k \quad = \quad \sum_k (-1)^k h_k\ .$$

By the identity $(*)$ mentioned at the beginning of this proof, we have $c_k - z_k = b_k$ and $z_k - b_{k+1} = h_k$, so that

$$\sum_k (-1)^k h_k \quad = \quad \sum_k (-1)^k (z_k - b_{k+1}) \quad = \quad \sum_k (-1)^k z_k \ - \ \sum_k (-1)^k b_{k+1} \quad =$$

$$\sum_r (-1)^r z_r \ + \ \sum_r (-1)^r b_r \quad = \quad \sum_k (-1)^k c_k$$

proving that the two sums in the theorem are equal.■