A nontrivial analogy between homology and cohomology

In the description of singular cohomology on pages 90–91 of the notes, several important analogies between singular homology and cohomology are discussed. However, there is one property of singular cohomology that differs notably from the corresponding property of singular homology, and it is the analog of the following normalization axiom:

(D.2) *If a space X is written as a union of its (pairwise disjoint) arc components X_α, then the inclusion maps $i_\alpha : X_\alpha \to X$ define an isomorphism from the (weak) direct sum $\oplus_\alpha\, H_*(X_\alpha)$ to $H_*(X)$.*

Here is the corresponding property of singular cohomology:

(D.2)* *If a space X is written as a union of its (pairwise disjoint) arc components X_α, then the inclusion maps $i_\alpha : X_\alpha \to X$ define an isomorphism from $H^*(X)$ to the direct* product $\prod_\alpha\, H^*(X_\alpha)$.

The reason for this difference is simple. The direct sum property for homology follows from a corresponding direct sum isomorphism of chain groups

$$\oplus_\alpha\, S_*(X_\alpha) \;\longrightarrow\; S_*(X)$$

(because each singular simplex lies in a unique arc component), and if we apply $\mathrm{Hom}(-,\pi)$ to this (where π is some abelian group) then we obtain an isomorphism

$$S^*(X;\pi) \;\longrightarrow\; \prod_\alpha\, S^*(X_\alpha;\pi)\ .$$

The final step is to note that if $\{C_\alpha\}$ is a family of cochain complexes, then the usual product defines a product $\prod\, C_\alpha$ in the category of cochain complexex, and the inclusion maps define an isomorphism from $H^*\left(\prod\, C_\alpha\right)$ to $\prod\, H^*(C_\alpha)$; the proof of this is left to the reader as an exercise.

Note that if X has finitely many arc components, then $H^*(X)$ is isomorphic to $\oplus_\alpha\, H^*(X_\alpha)$ because direct sums and products coincide when there are only finitely many summands/factors.