page 61, line ~~...~~ 21

If $K \subseteq G$ has finite index in G, let $\overline{G} = G/[G,G]$ and let $\overline{K}$ = image K in $\overline{G}$. Then $\overline{K}$ has finite index in $\overline{G}$.

Details By construction, $\overline{K}$ is $K \cdot [G,G]/[G,G]$, so if G is the union of the cosets $a_1 K, \ldots, a_m K$ then $\overline{G}$ is the union of cosets $a_j K \cdot [G,G]/[G,G]$. ■

p. 66 line -14

Simplicial approximation

Prop. 0 K = simplicial complex, P = underlying space, $\lambda : C_*(K) \longrightarrow S_*(P)$ the map previously called θ. Then the simplicial and singular barycentric subdivision maps are related by the following commutative diagram:

$$\begin{array}{ccc} C_*(K) & \xrightarrow{\beta} & C_*(BK) \\ \lambda \downarrow & & \downarrow \lambda \\ S_*(P) & \xrightarrow{\beta} & S_*(P). \end{array}$$

This follows from the constructions of the maps.

Simplicial Approximation Thm.

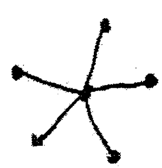
Let (P, K) and (Q, L) be simplicial complexes and let $f: P \to Q$ be continuous. Then $\exists r > 0$ and a simplicial map $g: B^r(K) \to L$ such that for all $x \in P$, if σ is a minimal simplex containing x, then $f(x) \in g[\sigma]$.

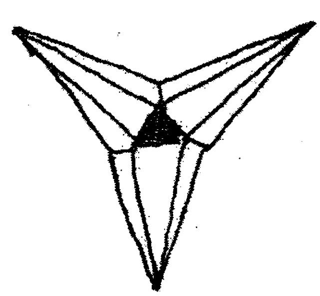
NEED SOME TERMINOLOGY

To some extend this corrects the definitions etc. on page 178 of Hatcher.

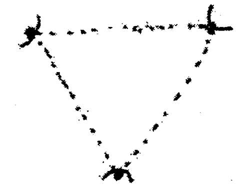
v vertex, σ simplex in K

The (closed) star $\operatorname{Star} \sigma$ = all simplices τ in K such that $\sigma \cap \tau \neq \emptyset$





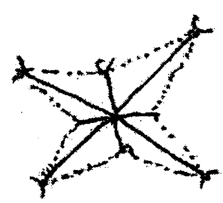
The <u>open simplex</u> $\mathring{\sigma} = \sigma - \partial\sigma$



[If σ has vertices v_i, then $\mathring{\sigma}$ = all $\sum t_i v_i$ s.t. $t_i > 0 \; \forall i$.

<u>Note</u> Every point lies in a unique open simplex.

The <u>open star</u> of a vertex v = all open simplices $\mathring{\sigma}$ s.t. $v \in \sigma$. $st(v)$ or Open star(v)



(v = vertex in the middle)

<u>Claim</u> Open star (v) is open in $|K|$ = underlying space of K.

In fact, $|K|$ - Open star (v) is the union of all (closed) simplices τ such that $v \notin \tau$.

Proof: Call the described set F. Then $x \in F \Rightarrow$ $x \in \tau$ where $v \notin \tau$. Let $\tau' \subseteq \tau$ such that $x \in \mathring{\tau}'$. Then v is also not a vertex of τ', so $x \notin$ Openstar(v). Conversely, $x \notin$ Openstar$(v) \Rightarrow x \in \mathring{\tau}$ where v is not a vertex of $\tau \Rightarrow x \in \tau \subseteq F$.

Key Lemma $v_0, \dots, v_q$ are vertices of a simplex in $K \Leftrightarrow \bigcap_i$ Openstar$(v_i) \neq \phi$.

Proof ($\Rightarrow$) Let $v_0, \dots, v_q$ be the vertices of σ, and let $y \in \mathring{\sigma}$. Then $y \in \bigcap$ Openstar(v_i) and in fact $\mathring{\sigma} \subseteq$ intersection.

($\Leftarrow$) Suppose $y \in \bigcap_i$ openstar(v_i) and let σ be the unique simplex such that $y \in \mathring{\sigma}$. Then for each i, v_i must be a vertex of σ, so there is a face of σ with vertices v_i (in fact, it's σ, but we don't need this).

NOTE In the lemma, duplications of v_i's are allowed.

<u>Proof of Thm.</u> Let $\mathcal{U}_0$ be the open covering of Q by sets Open star (w) where w runs through the vertices of L, and let $\mathcal{U} = f^{-1}\mathcal{U}_0$.

Using Lebesgue #s and barycentric subdivision, can find some $B^r K$ s.t. (i) each subset of P with diam $< \varepsilon$ lies in an element of $\mathcal{U}$, (ii) all simplices of $B^r K$ have diameter less than $\varepsilon/3$. Then each Star(v) has diam $\leq 2\varepsilon/3 < \varepsilon$, so $\exists$ vertex $g(v)$ in L s.t. $f[\text{Star}(v)] \subseteq$ Open star $g(v)$.

Define g : Vertices of $B^r K \longrightarrow$ Vertices of L using these choices.

CLAIM If $x \in v_0 \dots v_q$, where the latter is minimal, then $f(x) \in$ simplex with vertices $g(v_0) \dots g(v_q)$. $[x = \sum t_i v_i$, all $t_i > 0]$

This follows because $x \in \cap$ Openstar(v_i),

so $f(x) \in \bigcap_i f[\text{Openstar}(v_i)] \subseteq$

$\bigcap_i$ Openstar $g(v_i)$. This shows $g(v_i)$ are the vertices of a simplex, and the latter contains $f(x)$. Let $g(x) = \sum t_i g(v_i)$.

It follows that the image of the straight line homotopy

$$H(x,t) = t g(x) + (1-t) f(x)$$

lies in P so the latter defines a homotopy from f to g.

p. 67, line 10

The Lefschetz Fixed Point Theorem

(P, K) polyhedron

$f : P \to P$ cont.

The <u>Lefschetz number</u> $\Lambda(f) =$

$$\sum (-1)^k \text{ trace } f_{k*} : H_k(P;\mathbb{Q}) \to H_k(P;\mathbb{Q}).$$

<u>Claim</u> This is an integer.

<u>Idea</u> Look at the diagram

$$\begin{tikzcd}
H_k(P;\mathbb{Z})/\text{torsion} \arrow[r, "\subseteq"] \arrow[d, "f_{k*}^{\mathbb{Z}}"] & H_k(P;\mathbb{Q}) \arrow[d, "f_{k*}^{\mathbb{Q}}"] \\
H_k(P;\mathbb{Z}/\text{torsion}) \arrow[r, "\subseteq"] & H_k(P;\mathbb{Q})
\end{tikzcd}$$

and choose free generators for $H_k(P;\mathbb{Z})/\text{torsion}$ which yield a basis for $H_k(P;\mathbb{Q})$. This shows that a matrix for $f_*^{\mathbb{Q}}$ comes from an integral matrix for $f_*^{\mathbb{Z}}$.

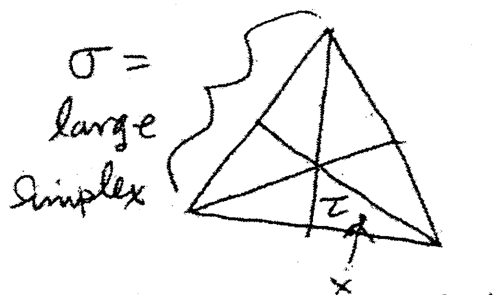
Hence each trace $f_{k*}^{\mathbb{Q}}$ is an integer.

THEOREM $\Lambda(f) \neq 0 \Rightarrow f$ has a fixed point.

PROOF. Suppose not, and let $\delta =$ min distance from $f(x)$ to x, so that $\delta > 0$ by compactness. Subdivide K into simplices of diameter $< \delta/4$. Then $f[\sigma] \cap \sigma$ is empty for all σ, and more generally $f[\mathrm{Star}\,\sigma] \cap \mathrm{Star}\,\sigma = \emptyset$ for all σ. Choose a simplicial approximation $g : B^r K \to K$ to f as in the Simplicial Approximation Thm.

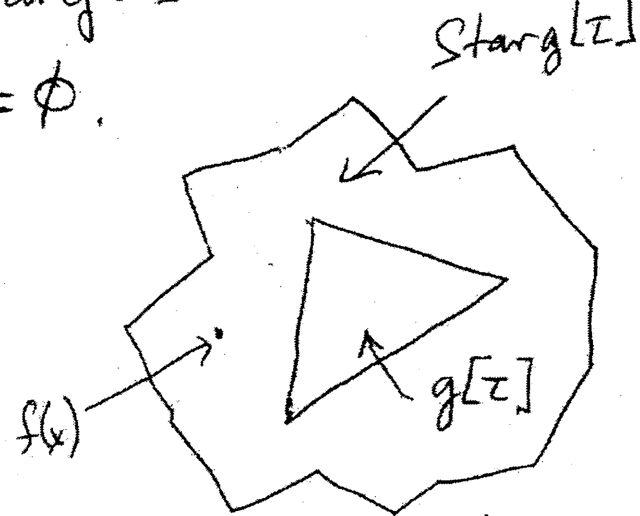
Then if τ is a simplex of $B^r(K)$ we have $f[\tau] \subseteq \mathrm{Star}\, g[\tau]$.

<u>**CLAIM**</u> $g[\tau] \cap \tau = \phi$.



τ in $B^r(K)$

$\tau \subseteq \sigma$ in K.



$g[\tau]$ in K.

$$x \in \tau \Rightarrow f(x) \in \mathrm{Star}\, g[\tau]$$

$$\text{and } d(x, f(x)) \geq \delta$$

$$y \in \tau \Rightarrow d(x,y) < \delta/4$$

$$z \in g[\tau] \Rightarrow f(x), z \in \mathrm{Star}\, g(\tau)$$

$$\Rightarrow d(f(x), z) < \delta/2.$$

It follows that $y \in \tau$, $z \in g[\tau] \Rightarrow$
$d(y,z) > \delta/4$.

In fact, we can say more: Given $\underset{\text{in } B^r(K)}{\tau} \subseteq \underset{\text{in } K}{\sigma}$ we have $\sigma \cap g(\tau) = \phi$.

Consider what this means for the chain map $C_*(K) \xrightarrow{\beta_r} C_*(B^r(K)) \xrightarrow{g_\#} C_*(K)$:

Let $\sigma = v_0 \dots v_q \in C_q(K)$ be a typical generator, so that $\beta_r(v_0 \dots v_q)$ lies in the chain subcomplex $C_*(B^r(v_0 \dots v_q))$, and consider the effect of $g_\#$ on a typical free generator of $C_q(B^r(v_0 \dots v_q))$.

If σ is the simplex with vertices $v_0 \dots v_q$, then $g_\#$ must take a typical free generator of $C_q(B^r(v_0 \dots v_q))$ into a chain subcomplex $C_*(\sigma') \subseteq C_*(K)$ such that $\sigma' \cap \sigma = \phi$. Therefore the image of $v_0 \dots v_q$ in $C_q(K)$ actually lies in some $C_q(K')$ where σ and K' are disjoint, and hence $g_\# \beta_r(v_0 \dots v_q)$ will lie in a subgroup of $C_q(K)$ consisting of chains whose $v_0 \dots v_q$-coordinates are zero. This means that the trace of $g_\# \beta_r : C_q(K) \longrightarrow C_q(K)$ is zero, and by the trace identity the same is true for $H_q(K;\mathbb{Q}) \xrightarrow{\beta_{r*}} H_q(B^r(K);\mathbb{Q}) \xrightarrow{g_*} H_q(K;\mathbb{Q})$. We know the latter corresponds to the singular homology map

$$g_* : H_q(P;\mathbb{Q}) \longrightarrow H_q(P;\mathbb{Q})$$

and hence its trace is also zero. Taking alternating sums, we see that $\Lambda(g)=0$. Finally, $f \simeq g \Rightarrow \Lambda(f)=\Lambda(g)$, and hence we also have $\Lambda(f)=0$.

To summarize, we have shown that if f has no fixed points, then $\Lambda(f)=0$.