page 9~~1~~, lines 10-11

Proof of Proposition 3.

NOTATION. If $e_0, \dots, e_r$ are the vertices of Δ_r, then $\mathrm{Front}_p(\Delta_r) =$ simplex $e_0 \dots e_p$ and $\mathrm{Back}_q(\Delta_r) =$ simplex $e_{r-q} \dots e_q$ (OK for all $r \geq p, q$).

(i) Naturality w.r.t. cont maps $h : (X, A) \to (Y, B)$.

Let $h : (X, A) \to (Y, B)$ be cont, let $f \in S^p(Y, B)$ and $g \in S^q(Y, B)$, and let $T : \Delta_{p+q} \to X$. Then

$$[h^*(f \cup g)](T) = f \cup g(h \circ T) \text{ by def.,}$$

and the latter is $f(\mathrm{Front}_p(hT)) \cdot g(\mathrm{Back}_q(hT)) =$
$h^\# f(\mathrm{Front}_p(T)) \cdot h^\# g(\mathrm{Back}_q(T)) = [h^\# f \cup h^\# g](T)$.

Hence the functions $h^*(f \cup g)$ and $h^\# f \cup h^\# g$ are equal (since we showed it for free generators).

This cochain lies in $S^{p+q}(X,A)$ because it vanishes if $T:\Delta_{p+q}\to A$.

$T:\Delta_{p+q}\to X$ as before

(ii) Bilinearity. $[(f_1+f_2)\cup g](T)=$
$[f_1+f_2](\mathrm{Front}_p(T))\cdot g(\mathrm{Back}_q(T))=$
$f_1(\mathrm{Front}_q(T)\cdot g(\mathrm{Back}_q(T))+f_2(\mathrm{Front}_q(T))\cdot g(\mathrm{Back}_q(T))$
$=[(f_1\cup g)+(f_2\cup g)](T)$, so the cochains are equal. Similarly to prove $f\cup(g_1+g_2)=(f\cup g_1)+(f\cup g_2)$.

(iii) Associativity. T as above $\Longrightarrow$
$[(f\cup g)\cup h](T)$ and $[f\cup(g\cup h)](T)$ both equal $f(T|e_0\ldots e_p))\cdot g(T|e_p\ldots e_{p+q})\cdot h(T|e_{p+q}\ldots e_{p+q+r})$.

(iv) Augmentation is a unit $T:\Delta_p\to X\Longrightarrow$

$$\left.\begin{aligned} f\cup\varepsilon(T)&=f(T)\cdot\underbrace{\varepsilon(T|\{e_p\})}_{1}=f(T)\\ \varepsilon\cup f(T)&=\underbrace{\varepsilon(T|\{e_0\})}_{1}\cdot f(T)=f(T)\end{aligned}\right\}\text{all } T.$$

page 92, lines 21-22

Proof that $\delta(f \cup g) = (\delta f) \cup g + (-1)^p f \cup (\delta g)$
where $f: S_p(X) \to \mathbb{D}$, $g: S_q(X) \to \mathbb{D}$.

Idea Prove they have the same value at a free generator $T: \Delta_{p+q+1} \to X$ of $S_{p+q+1}(X)$.

$\boxed{\text{Given } i_0 < \dots < i_r \text{, let } v_{i_0} \dots v_{i_r} = \text{restriction of } T \text{ to simplex w/ vertices } e_{i_j}}$

LHS $\delta(f \cup g)(T) =$

$$\sum_i (-1)^i f \cup g(v_0 \dots \hat{v_i} \dots v_{p+q+1}) =$$

$$\sum_{i \leq p} (-1)^i f(v_0 \dots \hat{v_i} \dots v_{p+1})\, g(v_{p+1} \dots v_{p+q+1}) +$$

$$\sum_{i \geq p+1} (-1)^i f(v_0 \dots v_p)\, g(v_p \dots \hat{v_i} \dots v_{p+q+1})$$

RHS Do pieces separately:

$(\delta f) \cup g(T) =$

$$\sum_{i=0}^{p+1} (-1)^i f(v_0 \dots \hat{v_i} \dots v_{p+1})\, g(v_{p+1} \dots v_{p+q+1})$$

<u>page 92, lines 21-22 continued</u>

$$(-1)^p f \cup \delta g\,(T) =$$

$$(-1)^p \sum_{i=p}^{p+q+1} (-1)^{i-p} f(v_0 \dots v_p)\, g(v_p \dots \hat{v_i} \dots v_{p+q+1})$$

If we subtract $\delta(f \cup g)(T)$ from $\delta f \cup g(T) + (-1)^p f \cup \delta g(T)$ we are left with the $p+1$ term in the sum for $\delta f \cup g(T)$ and the p term in the sum for $(-1)^p f \cup \delta g(T)$, which is

$$(-1)^{p+1} f(v_0 \dots v_p)\, g(v_{p+1} \dots v_{p+q+1}) +$$
$$(-1)^p f(v_0 \dots v_p)\, g(v_{p+1} \dots v_{p+q+1})$$

and since $(-1)^{p+1} + (-1)^p = 0$ these cancel each other. Hence the two expressions have the same value at T, which was what we wanted to prove.

page 93, lines 12-14

Comparison of simplicial and singular cup products.

Assertion (a) can be interpreted as saying that Proposition 3 also holds for simplicial cup products. The formal definition of the latter is $[f \cup g](v_0 \dots v_{p+q}) = f(v_0 \dots v_p) \cdot g(v_p \dots v_{p+q})$ and one can apply the arguments in the singular case provided we replace $T \mid e_i \dots e_j$ (singular) with $v_i \dots v_j$ (simplicial).

To prove assertion (b), let $\theta : C_*(K) \to S_*(P)$ be the map from simplicial to singular chains, and let $\Psi : S^*(P) \to C^*(P)$ be the dual cochain map.

Then $\Psi(f \cup g)(v_0 \dots v_{p+q}) = f \cup g(\theta(v_0 \dots v_{p+q})) =$

$f(\theta(v_0 \dots v_p)) \cdot g(\theta(v_p \dots v_{p+q})) =$

$\Psi f(v_0 \dots v_p) \cdot \Psi g(v_p \dots v_{p+q}) = [\Psi f \cup \Psi g](v_0 \dots v_{p+q})$.

page 97, line −5 to page 98, line 2

The formula discussed on these lines is

$$\langle \delta\alpha, y \rangle = \langle \alpha, \partial y \rangle$$

where all coefficients are in the field $\mathbb{F}$, with

$$\alpha \in H^{p-1}(A) \qquad \delta : H^{p-1}(A) \longrightarrow H^p(X,A)$$

$$y \in H_p(X,A) \qquad \partial : H_p(X,A) \longrightarrow H_{p-1}(A)$$

and $\langle\ ,\ \rangle$ = Kronecker index.

To verify this equation, use the definitions in terms of representative cycles and cocycles. Represent y by $c \in S_p(X)$ such that $dc \in S_{p-1}(A)$, and α by $g \in S^{p-1}(X)$ such that $\delta g = g \circ d \,|\, S_p(A) = 0$.

(Note that $S^{p-1}(X) \xrightarrow{\text{onto}} S^{p-1}(A)$, so every cocycle on A comes from a cochain on X). By definition, $\langle \alpha, \partial y \rangle$ is then equal to $g(dc) = g \circ d(c) = \delta g(c)$, which is equal to $\langle \delta\alpha, y \rangle$.

page 98, lines -12 to -11

Verification that $d_{n-1}^{A\otimes B} \circ d_n^{A\otimes B} = 0$.

Details: Since $(A\otimes B)_n$ is generated by classes $y\otimes z$ where $y\in A_p$ and $z\in B_{q=n-p}$ for some p, it suffices to check that the composite sends these elements to zero.

We have $d(y\otimes z) = dy\otimes z + (-1)^p y\otimes dz$, so $dd(y\otimes z) = d(dy\otimes z + (-1)^p y\otimes dz) =$ $ddy\otimes z + (-1)^{p-1} dy\otimes dz + (-1)^p dy\otimes dz +$ $(-1)^{p-1}(-1)^p y\otimes ddz$. The first and last terms are zero because $dd=0$, and the middle two terms cancel each other because $(-1)^p = -(-1)^{p-1}$. Thus $dd=0$ in $A\otimes B$, which is what we wanted to verify.

page 98, line -9

Proof of Proposition 1.

(i) Suppose $0 \to K_* \to A_* \to C_* \to 0$ is short exact and B_* is free in each dim. [As earlier on p. 99, assume all groups are zero in negative dims]. Note that if F is a free module and $0 \to K \to A \to C \to 0$ is short exact, then $F \cong \bigoplus_{\gamma \in \Gamma} \mathbb{D}_\gamma \Rightarrow$

$$0 \to K \otimes_{\mathbb{D}} F \to A \otimes_{\mathbb{D}} F \to C \otimes_{\mathbb{D}} F \to 0$$

is just a direct sum of the form

$$0 \to \bigoplus_\gamma K_\gamma \to \bigoplus_\gamma A_\gamma \to \bigoplus_\gamma C_\gamma \to 0$$

and is short exact (more generally, a direct sum of short exact sequences is short exact). Thus for each $n \geq 0$ the n-dimensional modules in the tensored chain complexes are given by the short exact sequence

<u>page 98, line -8, continued</u>

$$0 \to \bigoplus_{p\geq 0} K_p \otimes B_{n-p} \to \bigoplus_{p\geq 0} A_p \otimes B_{n-p} \to \bigoplus_{p\geq 0} C_p \otimes B_{n-p} \to 0$$

and hence $0 \to K_* \otimes B_* \to \underset{A_*\otimes B_*}{K_* \otimes A_*} \to \underset{C_*\otimes B_*}{K_* \otimes C_*} \to 0$ is short exact.

(ii) Now assume K_*, A_*, C_* are free in each dimension. Then for all p we have $A_p \cong C_p \oplus K_p$ and the n-dim modules in

$$0 \to K_* \otimes B_* \to A_* \otimes B_* \to C_* \otimes B_* \to 0$$

are given by the short exact sequence

$$0 \to \bigoplus_{p\geq 0} K_p \otimes B_{n-p} \to \bigoplus_{p\geq 0} (K_p \oplus C_p) \otimes B_{n-p} \to \bigoplus_{p\geq 0} C_p \otimes B_{n-p} \to 0$$

which is in fact a <u>split</u> short exact sequence. Although the chain complex sequence

$$0 \to K_* \otimes B_* \to A_* \otimes B_* \to C_* \otimes B_* \to 0$$

might not be (and usually is not) split, we can still conclude that it is short exact in the chain complex category.

page ~~100~~ 99, lines 1-9

Details for the proof of Proposition 2.

(i) If $x \in A_p$ and $y \in B_q$ are cycles, then $x \otimes y$ is also a cycle.

Proof. $dx = 0$ and $dy = 0 \Longrightarrow$
$d(x \otimes y) = \underbrace{dx}_{0} \otimes y + (-1)^p x \otimes \underbrace{dy}_{0} = 0 + 0 = 0$.

(ii) If $x, x' \in A_p$ and $y, y' \in B_q$ are cycles such that $x - x' = dw$ and $y - y' = dz$ for some w and z, then $[x \otimes y] = [x' \otimes y']$ in $H_{p+q}(A \otimes B)$.

Proof. $d(w \otimes y) = (x - x') \otimes y + (-1)^{p+1} w \otimes \underbrace{dy}_{0} =$
$x \otimes y - x' \otimes y$, so $[x \otimes y] = [x' \otimes y]$ in $H_*(A \otimes B)$. Likewise, $d(x' \otimes z) =$
$\underbrace{dx'}_{0} \otimes z + (-1)^p x' \otimes (y - y') = (-1)^p (x' \otimes y - x' \otimes y')$,
so $[x' \otimes y] = [x' \otimes y']$ in $H_*(A \otimes B)$.

page ~~10~~ 99, lines 1-9, continued

The preceding shows that the map
$[x]\otimes[y] \xrightarrow[\text{``}\otimes\text{''}]{\mu_*} [x\otimes y]$ is well-defined.

(iii) The map ↑ is bilinear.

This is easy given the well-definition of
μ_*: $([x_1]+[x_2])\otimes[y] = [x_1+x_2]\otimes[y] =$
$[(x_1+x_2)\otimes y] = [(x_1\otimes y)+(x_2\otimes y)] = ([x_1]\otimes[y]) + ([x_2]\otimes[y])$,
and similarly we have $[x]\otimes([y_1]+[y_2]) =$
$([x]\otimes[y_1]) + ([x]\otimes[y_2])$.

Furthermore, if $c\in\mathbb{D}$ then $c\cdot([x]\otimes[y]) =$
$c[x\otimes y] = [c(x\otimes y)] = [(cx)\otimes y] = [cx]\otimes[y]$,
and similarly we have $c([x]\otimes[y]) = [x]\otimes[cy]$
(recall that $c(a\otimes b) = (ca)\otimes b = a\otimes cb$ in
the tensor product $M\otimes_{\mathbb{D}}N$, where $\mathbb{D}$ is a
commutative ring with unit).

page 103, line 5

Verification that the Alexander-Whitney map (on line 4) is a chain map.

As in many other instances, we start with the universal example where $T: \Delta_n \to \Delta_n \times \Delta_n$ is the diagonal map, and we write this in the form $(x_0 \cdots x_n; y_0 \cdots y_n)$. Then

$$\underset{\text{1st coord}}{\uparrow} \quad \underset{\text{2nd coord.}}{\uparrow}$$

$$d_n \Psi(x_0 \cdots x_n; y_0 \cdots y_n) = d\sum_{p=0}^{n} x_0 \cdots x_p \otimes y_p \cdots y_n =$$

$$\sum_{p=0}^{n} d(x_0 \cdots x_p \otimes y_p \cdots y_n) =$$

$$\sum_{p=0}^{n} d(x_0 \cdots x_p) \otimes y_p \cdots y_n + (-1)^p \sum_{p=0}^{n} x_0 \cdots x_p \otimes d(y_p \cdots y_n)$$

$$= \sum_{p=0}^{n} \sum_{i=0}^{p} (-1)^i x_0 \cdots \overset{\text{OMIT}}{(x_i)} \cdots x_p \otimes y_p \cdots y_n +$$

$$\sum_{p=0}^{n} \sum_{i=p}^{n} x_0 \cdots x_p \otimes y_p \cdots \overset{\text{OMIT}}{(y_i)} \cdots y_n . \ (-1)^i$$

page 103, line 5, continued

On the other hand,

$$\psi\, d_n\,(x_0 \cdots x_n;\, y_0 \cdots y_n) = \psi \sum_{j=0}^{n} (-1)^j (x_0 \cdots (x_j) \cdots x_n;\, y_0 \cdots (y_j) \cdots y_n)$$

$$= \sum_{j=0}^{n} (-1)^j \Big[\sum_{p \le j} x_0 \cdots x_p \otimes y_p \cdots \overset{\text{OMIT}}{(y_j)} \cdots y_n \; +$$

$$\sum_{p \ge j} x_0 \cdots \overset{\text{OMIT}}{(x_j)} \cdots x_p \otimes y_p \cdots y_n \Big] . \; I_6$$

we subtract the second expression from the first, we are left with

$$\sum_{p=0}^{n} (-1)^p\, x_0 \cdots \overset{\text{OMIT}}{(x_p)} \otimes y_p \cdots y_n \; +$$

$$\sum_{p=0}^{n} (-1)^{p-1} x_0 \cdots x_{p-1} \otimes \overset{\text{OMIT}}{\overbrace{y_{p-1}}}\, y_p \cdots y_n$$

which is zero. Hence $\psi d_n = d_n \psi$ on $(x_0 \cdots x_n;\, y_0 \cdots y_n)$.

page 10~~4~~3, line 5, continued

<u>General case</u> $T : \Delta_n \to X \times Y$, $T = (T_X, T_Y)$

Note that $T_\# \Psi = \Psi T_\#$ by construction, for both evaluated at $\underset{= U}{(x_0 \cdots x_n; y_0 \cdots y_n)}$ yield

$$\sum_{p=0}^{n} \mathrm{Front}_p(T_X) \otimes \mathrm{Back}_{n-p}(T_Y).$$

Then $\Psi dT = \Psi d T_\#(U) = \Psi T_\# d(U) =$

$T_\# \Psi d(U) \underset{\text{PREV}}{=} T_\# d\Psi(U) = d T_\# \Psi(U) =$

$d\Psi T_\#(U) = d\Psi(T)$.

page 105, line 16

$$d\varphi(F_X \otimes B_Y) = d(F_X \times B_Y)_\# \varphi(id_p \otimes id_{n-p}) =$$

$$(F_X \times B_Y)_\# \varphi d(id_p \otimes id_{n-p}) \underset{\text{this!}}{\overset{\text{check}}{=}}$$

$$\varphi \circ (F_X \times B_Y)_\# d(id_p \otimes id_{n-p}) =$$

$$\varphi d(F_X \times B_Y)_\# (id_p \otimes id_{n-p}) = \varphi d(F_X \otimes B_Y)$$

Therefore d is a chain map.

page 103 lines 20–21

Relative cross products for closed subsets.

$A \subseteq U \subseteq X$, $B \subseteq V \subseteq Y$ s.t.
closed open, closed open

A is a def. retract of U & B is a def. retract of B.

Then we have

$$\begin{tikzcd}
H^*(X,A) \otimes H^*(Y,B) \arrow[drr, dashed] & & H^*(X,U) \otimes H^*(Y,B) \arrow[ll, "\cong"'] \arrow[d, "\times"', "\text{prod}"] \\
& & H^*(X \times Y, U \times Y \cup X \times V) \arrow[d, "(\text{inclusion})^*"] \\
& & H^*(X \times Y, A \times Y \cup X \times B)
\end{tikzcd}$$

Note If (K, L) is a simplicial complex pair with underlying space pair $(|K|, |L|)$, then $|L|$ is a deformation retract of an open neighborhood W in $|K|$

page ~~107~~ 10?, lines 12-13

WORK ON THE CHAIN/COCHAIN LEVEL (see notes)

Proof of Proposition 14.

(i) Let $c \in S_q(X)$ be a chain. Then

$$\varepsilon_X \cap c = \underbrace{\varepsilon_X(T|\{e_0\})}_{=1} \cdot \underbrace{\mathrm{Back}_q(c)}_{=c} = c$$

and $c \cap \varepsilon_X = \mathrm{Front}_q(c) \cdot \varepsilon_X(T|\{e_q\}) = c$ likewise.

[$\mathrm{Back}_q = \mathrm{id}$ on q-simplices!]

(ii) Check that both cochains $(f \cup g) \cap c$ and $f \cap (g \cap c)$ are equal to $f(T|e_0 \ldots e_q) \cdot g(T|e_q \ldots e_p) \cdot \mathrm{Back}_r\, c$

where $r = n - p - q$.

(iii) Let $g \in S^p(Y)$ and $c \in S_n(X)$. Then

$g \cap f_\# c = g(\mathrm{Front}_p(f_\# c)) \cdot \mathrm{Back}_{n-p}(f_\# c) =$

$g(f_\# \mathrm{Front}_p\, c) \cdot f_\# \mathrm{Back}_{n-p}(c) =$

$f^\# g(\mathrm{Front}_p\, c) \cdot f_\#(\mathrm{Back}_{n-p}(c)) \xrightarrow[\text{module hom}]{f_\# \text{ is a}}$

page ~~107~~ 10, lines 12-13 continued

$$f_\#\left(f_g^\#(\mathrm{Front}_p c)\cdot \mathrm{Back}_{n-p}(c)\right) =$$

$$f_\#\left(f_g^\# \cap c\right).$$

By the discussion preceding the statement of Prop 14, the corresponding identities in homology and cohomology follow directly from these.

page 108, lines −5 to −4

Proof that

(1) the Alexander Whitney map $S_*(X) \longrightarrow S_*(X) \otimes S_*(X)$

is coassociative,

(2) the Alexander-Whitney map is functorial with respect to cont. maps $f: X \to Y$.

---

(1) Both $(\Psi \otimes id) \circ \Psi$ and $(id \otimes \Psi) \circ \Psi$ send

$$T: \Delta_n \to X \text{ to } \sum_{0 \le s \le t \le n} 2 (T|e_0 \dots e_s) \otimes (T|e_s \dots e_t) \otimes (T|e_t \dots e_n).$$

(2) The goal is to verify that the diagram

$$\begin{array}{ccc} S_*(X) & \xrightarrow{\Psi_X} & S_*(X) \otimes S_*(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow {\scriptstyle f_* \otimes f_*} \\ S_*(Y) & \xrightarrow{\Psi_Y} & S_*(Y) \otimes S_*(Y) \end{array}$$

commutes.

If we apply both/either of these composites to $T: \Delta_n \to X$, the result is

$$\sum \mathrm{Front}_p(f \circ T) \otimes \mathrm{Back}_{n-p}(f \circ T).$$

page ~~110~~ 109, line 10

Singular augmentations are co-units

(1) $S_*(X) \xrightarrow{\Psi} S_*(X) \otimes S_*(X) \longrightarrow \mathbb{D} \otimes S_*(X) \cong S_*(X)$.

Evaluate on $T : \Delta_n \to X$:

$$T \longmapsto \sum_p \mathrm{Front}_p(T) \otimes \mathrm{Back}_{n-p}(T) \longmapsto \varepsilon(\mathrm{Front}_0(T)) \otimes T \approx T$$

$\Rightarrow$ composite is the identity.

(2) $S_*(X) \xrightarrow{\Psi} S_*(X) \otimes S_*(X) \longrightarrow S_*(X) \otimes \mathbb{D} \cong S_*(X)$

Evaluate on $T : \Delta_n \to X$:

$$T \longmapsto \sum_p \mathrm{Front}_p(T) \otimes \mathrm{Back}_{n-p}(T) \longmapsto T \otimes \varepsilon(\mathrm{Back}_0(T)) \approx T$$

$\Rightarrow$ composite is the identity.

page 109 lines 19-20

Verification that $\tau : A_* \otimes B_* \to B_* \otimes A_*$ is a chain map.

It suffices to show $d\tau(a\otimes b) = \tau d(a\otimes b)$ for $a \in A_p$, $b \in B_q$.

$d\tau(a\otimes b) = d[(-1)^{pq} b\otimes a] =$

$(-1)^{pq}[db\otimes a + (-1)^{p} b\otimes da] =$

$(-1)^{pq}[db\otimes a + (-1)^{q\,(p+1)q} b\otimes da$. Also,

$\tau d(a\otimes b) = \tau(da\otimes b + (-1)^p a\otimes db) =$

$(-1)^{(p-1)q} b\otimes da + (-1)^{(p-1)q}(-1)^p da\otimes b =$

★ $\updownarrow$

$(-1)^{(p+1)q} b\otimes da + (-1)^{pq} da\otimes b$,

so $d\tau(a\otimes b) = \tau d(a\otimes b)$.

★: $(-1)^{(p-1)q} = (-1)^{(p-1)q}\cdot \underbrace{(-1)^{2q}}_{1} = (-1)^{(p+1)q}$

page 112, lines 7–8

This follows immediately from Theorem IV. 3.10 (p. 103).

page 111, line −1

Computation of $\left(\sum_{k=1}^{r} u_k\right)^r$.

One can use the multinomial theorem to expand this because the u_k's commute with each other. Furthermore, $u_k^2 = 0 \Rightarrow$ the only nonzero terms have the form $u_1 \cdots u_r$, and the multinomial coefficient is $\dfrac{r!}{1!\,1!\cdots 1!} = r!$

so that $\left(\sum_{k=1}^{r} u_k\right)^r = r! \prod_{k=1}^{r} u_k$.