page 97, lines 13-14

An important step in the argument is the following:

CLAIM Let W be a vector space over the field $\mathbb{F}$, and let $W_1 \subseteq W_2 \subseteq W$ be subspaces. If $W_i^{\perp}$ is the annihilator of W_i in W^*, then there is a canonical isomorphism $(W_2/W_1)^* \to W_1^{\perp}/W_2^{\perp}$.

Verification of Claim: Note that $(W_2/W_1)^* \cong$ all linear functionals $f: W_2 \to \mathbb{F}$ such that $f|W_1 = 0$. Choose a subspace $U \subseteq W$ s.t. $U \oplus W_2 = W$, and extend f to $\bar{f} \in W^*$ by setting $\bar{f}|U = 0$. By construction the map $(W_2/W_1)^* \to W^*$ sending f to $\bar{f}$ is $\mathbb{F}$-linear, and its image is contained in $W_1^{\perp}$. Therefore we obtain a map $(W_2/W_1)^* \to W_1^{\perp}/W_2^{\perp}$ by composing with $W_1^{\perp} \to W_1^{\perp}/W_2^{\perp}$.

We need to prove this is an isomorphism.

<u>ONTO</u> Since $W_1^+ \longrightarrow W_1^+/W_2^+$ is onto it is enough to show that $(W_2/W_1)^* \longrightarrow W_1^+$ is onto, and this follows because $h: W \to \mathbb{F}$ with $h|W_1 = 0$ is the image of h' in the diagram below:

$$\begin{tikzcd} W_2 \arrow[r] \arrow[dr, dashed, "\text{QUOTIENT}"'] & W \arrow[r, "h"] & \mathbb{F} \\ & W_2/W_1 \arrow[ur, "h'"'] & \end{tikzcd}$$

(recall $h|W_1 = 0$)

<u>ONE-TO-ONE</u> Given $f: W_2 \to \mathbb{F}$ with $f|W_1 = 0$, suppose that its image in W_1^+/W_2^+ is trivial. This means that $\bar{f} \in W_2^+$, or equivalently $\bar{f}|W_2 = f|W_2 = 0$. Since W_2 is the domain of definition for f, we must have $f = 0$.

Combining these, we obtain the desired isomorphism

$$(W_2/W_1)^* \longrightarrow W_1^+/W_2^+. \blacksquare$$