Details of proofs for several results on differential forms

The attached pages from the Third Edition of Rudin, ***Principles of Mathematical Analysis*** (pp. **262 – 280**), contain proofs or examples for the following, which are either in the main notes or closely related to them:

1. **Theorem V.0.6** on change of variables for differential forms.
2. An example of a closed 2–form on $\mathbb{R}^2 - \{0\}$ which is not exact (this corresponds to a vector field $\mathbf{F}$ such the divergence of $\mathbf{F}$ is zero but $\mathbf{F}$ is not the curl of some vector field on the given open set (see Example 10.37 on page 277 of Rudin, which is page 17 of this document).
3. **Proposition V.1.1** on the change of variables rule for integrating differential forms over smooth singular chains ("with respect to the bilinear map defined by taking integrals, the pullback for differential forms associated to a smooth function f is adjoint to the induced map for smooth singular chains").
4. **Theorem V.1.2**, which is the general version of Stokes' Theorem for integrals of differential forms over smooth singular chains ("with respect to the bilinear map defined by taking integrals, the exterior derivative for differential forms is adjoint to the boundary map for smooth singular chains").

by (42). Hence

$$d(\omega \wedge \lambda) = (df \wedge dx_I) \wedge (g\, dx_J) + (-1)^k (f\, dx_I) \wedge (dg \wedge dx_J)$$
$$= (d\omega) \wedge \lambda + (-1)^k \omega \wedge d\lambda,$$

which proves (*a*).

Note that the associative law (58) was used freely.

Let us prove (*b*) first for a 0-form $f \in \mathscr{C}''$:

$$d^2 f = d\left(\sum_{j=1}^{n} (D_j f)(\mathbf{x})\, dx_j\right)$$
$$= \sum_{j=1}^{n} d(D_j f) \wedge dx_j$$
$$= \sum_{i,j=1}^{n} (D_{ij} f)(\mathbf{x})\, dx_i \wedge dx_j .$$

Since $D_{ij} f = D_{ji} f$ (Theorem 9.41) and $dx_i \wedge dx_j = -dx_j \wedge dx_i$, we see that $d^2 f = 0$.

If $\omega = f\, dx_I$, as in (64), then $d\omega = (df) \wedge dx_I$. By (60), $d(dx_I) = 0$. Hence (63) shows that

$$d^2 \omega = (d^2 f) \wedge dx_I = 0.$$

10.21 Change of variables Suppose E is an open set in R^n, T is a $\mathscr{C}'$-mapping of E into an open set $V \subset R^m$, and ω is a k-form in V, whose standard presentation is

$$\omega = \sum_I b_I(\mathbf{y})\, dy_I . \tag{65}$$

(We use $\mathbf{y}$ for points of V, $\mathbf{x}$ for points of E.)

Let $t_1, \ldots, t_m$ be the components of T: If

$$\mathbf{y} = (y_1, \ldots, y_m) = T(\mathbf{x})$$

then $y_i = t_i(\mathbf{x})$. As in (59),

$$dt_i = \sum_{j=1}^{n} (D_j t_i)(\mathbf{x})\, dx_j \qquad (1 \le i \le m). \tag{66}$$

Thus each dt_i is a 1-form in E.

The mapping T transforms ω into a k-form ω_T in E, whose definition is

$$\omega_T = \sum_I b_I(T(\mathbf{x}))\, dt_{i_1} \wedge \cdots \wedge dt_{i_k} . \tag{67}$$

In each summand of (67), $I = \{i_1, \ldots, i_k\}$ is an increasing k-index.

Our next theorem shows that addition, multiplication, and differentiation of forms are defined in such a way that they commute with changes of variables.

10.22 Theorem *With E and T as in Sec.* 10.21, *let ω and λ be k- and m-forms in V, respectively. Then*

(*a*) $(\omega + \lambda)_T = \omega_T + \lambda_T$ *if* $k = m$;
(*b*) $(\omega \wedge \lambda)_T = \omega_T \wedge \lambda_T$;
(*c*) $d(\omega_T) = (d\omega)_T$ *if ω is of class $\mathscr{C}'$ and T is of class $\mathscr{C}''$.*

Proof Part (*a*) follows immediately from the definitions. Part (*b*) is almost as obvious, once we realize that

(68) $$(dy_{i_1} \wedge \cdots \wedge dy_{i_r})_T = dt_{i_1} \wedge \cdots \wedge dt_{i_r}$$

regardless of whether $\{i_1, \ldots, i_r\}$ is increasing or not; (68) holds because the same number of minus signs are needed on each side of (68) to produce increasing rearrangements.

We turn to the proof of (*c*). If f is a 0-form of class $\mathscr{C}'$ in V, then

$$f_T(\mathbf{x}) = f(T(\mathbf{x})), \qquad df = \sum_i (D_i f)(\mathbf{y})\, dy_i.$$

By the chain rule, it follows that

(69) $$\begin{aligned} d(f_T) &= \sum_j (D_j f_T)(\mathbf{x})\, dx_j \\ &= \sum_j \sum_i (D_i f)(T(\mathbf{x}))(D_j t_i)(\mathbf{x})\, dx_j \\ &= \sum_i (D_i f)(T(\mathbf{x}))\, dt_i \\ &= (df)_T. \end{aligned}$$

If $dy_I = dy_{i_1} \wedge \cdots \wedge dy_{i_k}$, then $(dy_I)_T = dt_{i_1} \wedge \cdots \wedge dt_{i_k}$, and Theorem 10.20 shows that

(70) $$d((dy_I)_T) = 0.$$

(This is where the assumption $T \in \mathscr{C}''$ is used.)

Assume now that $\omega = f\, dy_I$. Then

$$\omega_T = f_T(\mathbf{x})\, (dy_I)_T$$

and the preceding calculations lead to

$$\begin{aligned} d(\omega_T) &= d(f_T) \wedge (dy_I)_T = (df)_T \wedge (dy_I)_T \\ &= ((df) \wedge dy_I)_T = (d\omega)_T. \end{aligned}$$

The first equality holds by (63) and (70), the second by (69), the third by part (*b*), and the last by the definition of $d\omega$.

The general case of (*c*) follows from the special case just proved, if we apply (*a*). This completes the proof.

Our next objective is Theorem 10.25. This will follow directly from two other important transformation properties of differential forms, which we state first.

10.23 Theorem *Suppose T is a $\mathscr{C}'$-mapping of an open set $E \subset R^n$ into an open set $V \subset R^m$, S is a $\mathscr{C}'$-mapping of V into an open set $W \subset R^p$, and ω is a k-form in W, so that ω_S is a k-form in V and both $(\omega_S)_T$ and ω_{ST} are k-forms in E, where ST is defined by $(ST)(\mathbf{x}) = S(T(\mathbf{x}))$. Then*

$$(\omega_S)_T = \omega_{ST}. \tag{71}$$

Proof If ω and λ are forms in W, Theorem 10.22 shows that

$$((\omega \wedge \lambda)_S)_T = (\omega_S \wedge \lambda_S)_T = (\omega_S)_T \wedge (\lambda_S)_T$$

and

$$(\omega \wedge \lambda)_{ST} = \omega_{ST} \wedge \lambda_{ST}.$$

Thus if (71) holds for ω and for λ, it follows that (71) also holds for $\omega \wedge \lambda$. Since every form can be built up from 0-forms and 1-forms by addition and multiplication, and since (71) is trivial for 0-forms, it is enough to prove (71) in the case $\omega = dz_q$, $q = 1, \ldots, p$. (We denote the points of E, V, W by $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, respectively.)

Let $t_1, \ldots, t_m$ be the components of T, let $s_1, \ldots, s_p$ be the components of S, and let $r_1, \ldots, r_p$ be the components of ST. If $\omega = dz_q$, then

$$\omega_S = ds_q = \sum_j (D_j s_q)(\mathbf{y})\, dy_j,$$

so that the chain rule implies

$$\begin{aligned}(\omega_S)_T &= \sum_j (D_j s_q)(T(\mathbf{x}))\, dt_j \\ &= \sum_j (D_j s_q)(T(\mathbf{x})) \sum_i (D_i t_j)(\mathbf{x})\, dx_i \\ &= \sum_i (D_i r_q)(\mathbf{x})\, dx_i = dr_q = \omega_{ST}.\end{aligned}$$

10.24 Theorem *Suppose ω is a k-form in an open set $E \subset R^n$, Φ is a k-surface in E, with parameter domain $D \subset R^k$, and Δ is the k-surface in R^k, with parameter domain D, defined by $\Delta(\mathbf{u}) = \mathbf{u}\,(\mathbf{u} \in D)$. Then*

$$\int_\Phi \omega = \int_\Delta \omega_\Phi.$$

Proof We need only consider the case

$$\omega = a(\mathbf{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

If $\phi_1, \ldots, \phi_n$ are the components of Φ, then

$$\omega_\Phi = a(\Phi(\mathbf{u}))\, d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k}.$$

The theorem will follow if we can show that

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} = J(\mathbf{u})\, du_1 \wedge \cdots \wedge du_k, \tag{72}$$

where

$$J(\mathbf{u}) = \frac{\partial(x_{i_1}, \ldots, x_{i_k})}{\partial(u_1, \ldots, u_k)},$$

since (72) implies

$$\int_\Phi \omega = \int_D a(\Phi(\mathbf{u}))J(\mathbf{u})\, d\mathbf{u}$$

$$= \int_\Delta a(\Phi(\mathbf{u}))J(\mathbf{u})\, du_1 \wedge \cdots \wedge du_k = \int_\Delta \omega_\Phi.$$

Let $[A]$ be the k by k matrix with entries

$$\alpha(p, q) = (D_q \phi_{i_p})(\mathbf{u}) \qquad (p, q = 1, \ldots, k).$$

Then

$$d\phi_{i_p} = \sum_q \alpha(p, q)\, du_q$$

so that

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} = \sum \alpha(1, q_1) \cdots \alpha(k, q_k)\, du_{q_1} \wedge \cdots \wedge du_{q_k}.$$

In this last sum, $q_1, \ldots, q_k$ range independently over $1, \ldots, k$. The anti-commutative relation (42) implies that

$$du_{q_1} \wedge \cdots \wedge du_{q_k} = s(q_1, \ldots, q_k)\, du_1 \wedge \cdots \wedge du_k,$$

where s is as in Definition 9.33; applying this definition, we see that

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} = \det[A]\, du_1 \wedge \cdots \wedge du_k;$$

and since $J(\mathbf{u}) = \det[A]$, (72) is proved.

The final result of this section combines the two preceding theorems.

10.25 Theorem *Suppose T is a $\mathscr{C}'$-mapping of an open set $E \subset R^n$ into an open set $V \subset R^m$, Φ is a k-surface in E, and ω is a k-form in V.*

Then

$$\int_{T\Phi} \omega = \int_\Phi \omega_T.$$

Proof Let D be the parameter domain of Φ (hence also of $T\Phi$) and define Δ as in Theorem 10.24.

Then

$$\int_{T\Phi} \omega = \int_{\Delta} \omega_{T\Phi} = \int_{\Delta} (\omega_T)_{\Phi} = \int_{\Phi} \omega_T .$$

The first of these equalities is Theorem 10.24, applied to $T\Phi$ in place of Φ. The second follows from Theorem 10.23. The third is Theorem 10.24, with ω_T in place of ω.

SIMPLEXES AND CHAINS

10.26 Affine simplexes A mapping $\mathbf{f}$ that carries a vector space X into a vector space Y is said to be *affine* if $\mathbf{f} - \mathbf{f}(\mathbf{0})$ is linear. In other words, the requirement is that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{0}) + A\mathbf{x} \tag{73}$$

for some $A \in L(X, Y)$.

An affine mapping of R^k into R^n is thus determined if we know $\mathbf{f}(\mathbf{0})$ and $\mathbf{f}(\mathbf{e}_i)$ for $1 \le i \le k$; as usual, $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ is the standard basis of R^k.

We define the *standard simplex* Q^k to be the set of all $\mathbf{u} \in R^k$ of the form

$$\mathbf{u} = \sum_{i=1}^{k} \alpha_i \mathbf{e}_i \tag{74}$$

such that $\alpha_i \ge 0$ for $i = 1, \ldots, k$ and $\Sigma\alpha_i \le 1$.

Assume now that $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_k$ are points of R^n. The *oriented affine k-simplex*

$$\sigma = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_k] \tag{75}$$

is defined to be the k-surface in R^n with parameter domain Q^k which is given by the affine mapping

$$\sigma(\alpha_1\mathbf{e}_1 + \cdots + \alpha_k \mathbf{e}_k) = \mathbf{p}_0 + \sum_{i=1}^{k} \alpha_i(\mathbf{p}_i - \mathbf{p}_0). \tag{76}$$

Note that σ is characterized by

$$\sigma(\mathbf{0}) = \mathbf{p}_0, \qquad \sigma(\mathbf{e}_i) = \mathbf{p}_i \qquad (\text{for } 1 \le i \le k), \tag{77}$$

and that

$$\sigma(\mathbf{u}) = \mathbf{p}_0 + A\mathbf{u} \qquad (\mathbf{u} \in Q^k) \tag{78}$$

where $A \in L(R^k, R^n)$ and $A\mathbf{e}_i = \mathbf{p}_i - \mathbf{p}_0$ for $1 \le i \le k$.

We call σ *oriented* to emphasize that the ordering of the vertices $\mathbf{p}_0, \ldots, \mathbf{p}_k$ is taken into account. If

$$\bar{\sigma} = [p_{i_0}, p_{i_1}, \ldots, p_{i_k}], \tag{79}$$

where $\{i_0, i_1, \ldots, i_k\}$ is a permutation of the ordered set $\{0, 1, \ldots, k\}$, we adopt the notation

$$\bar{\sigma} = s(i_0, i_1, \ldots, i_k)\sigma, \tag{80}$$

where s is the function defined in Definition 9.33. Thus $\bar{\sigma} = \pm\sigma$, depending on whether $s = 1$ or $s = -1$. Strictly speaking, having adopted (75) and (76) as the definition of σ, we should not write $\bar{\sigma} = \sigma$ unless $i_0 = 0, \ldots, i_k = k$, even if $s(i_0, \ldots, i_k) = 1$; what we have here is an equivalence relation, not an equality. However, for our purposes the notation is justified by Theorem 10.27.

If $\bar{\sigma} = \varepsilon\sigma$ (using the above convention) and if $\varepsilon = 1$, we say that $\bar{\sigma}$ and σ have the *same orientation*; if $\varepsilon = -1$, $\bar{\sigma}$ and σ are said to have *opposite orientations*. Note that we have not defined what we mean by the "orientation of a simplex." What we have defined is a relation between pairs of simplexes having the same set of vertices, the relation being that of "having the same orientation."

There is, however, one situation where the orientation of a simplex can be defined in a natural way. This happens when $n = k$ and when the vectors $\mathbf{p}_i - \mathbf{p}_0$ $(1 \leq i \leq k)$ are *independent*. In that case, the linear transformation A that appears in (78) is invertible, and its determinant (which is the same as the Jacobian of σ) is not 0. Then σ is said to be *positively* (or *negatively*) oriented if $\det A$ is positive (or negative). In particular, the simplex $[\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_k]$ in R^k, given by the identity mapping, has positive orientation.

So far we have assumed that $k \geq 1$. An *oriented 0-simplex* is defined to be a point with a sign attached. We write $\sigma = +\mathbf{p}_0$ or $\sigma = -\mathbf{p}_0$. If $\sigma = \varepsilon\mathbf{p}_0$ $(\varepsilon = \pm 1)$ and if f is a 0-form (i.e., a real function), we define

$$\int_\sigma f = \varepsilon f(p_0).$$

10.27 Theorem *If σ is an oriented rectilinear k-simplex in an open set $E \subset R^n$ and if $\bar{\sigma} = \varepsilon\sigma$ then*

$$\int_{\bar{\sigma}} \omega = \varepsilon \int_\sigma \omega \tag{81}$$

for every k-form ω in E.

Proof For $k = 0$, (81) follows from the preceding definition. So we assume $k \geq 1$ and assume that σ is given by (75).

Suppose $1 \le j \le k$, and suppose $\bar{\sigma}$ is obtained from σ by interchanging $\mathbf{p}_0$ and $\mathbf{p}_j$. Then $\varepsilon = -1$, and

$$\bar{\sigma}(\mathbf{u}) = \mathbf{p}_j + B\mathbf{u} \qquad (\mathbf{u} \in Q^k),$$

where B is the linear mapping of R^k into R^n defined by $B\mathbf{e}_j = \mathbf{p}_0 - \mathbf{p}_j$, $B\mathbf{e}_i = \mathbf{p}_i - \mathbf{p}_j$ if $i \neq j$. If we write $A\mathbf{e}_i = \mathbf{x}_i$ $(1 \le i \le k)$, where A is given by (78), the column vectors of B (that is, the vectors $B\mathbf{e}_i$) are

$$\mathbf{x}_1 - \mathbf{x}_j, \ldots, \mathbf{x}_{j-1} - \mathbf{x}_j, -\mathbf{x}_j, \mathbf{x}_{j+1} - \mathbf{x}_j, \ldots, \mathbf{x}_k - \mathbf{x}_j.$$

If we subtract the jth column from each of the others, none of the determinants in (35) are affected, and we obtain columns $\mathbf{x}_1, \ldots, \mathbf{x}_{j-1}, -\mathbf{x}_j, \mathbf{x}_{j+1}, \ldots, \mathbf{x}_k$. These differ from those of A only in the sign of the jth column. Hence (81) holds for this case.

Suppose next that $0 < i < j \le k$ and that $\bar{\sigma}$ is obtained from σ by interchanging $\mathbf{p}_i$ and $\mathbf{p}_j$. Then $\bar{\sigma}(\mathbf{u}) = \mathbf{p}_0 + C\mathbf{u}$, where C has the same columns as A, except that the ith and jth columns have been interchanged. This again implies that (81) holds, since $\varepsilon = -1$.

The general case follows, since every permutation of $\{0, 1, \ldots, k\}$ is a composition of the special cases we have just dealt with.

10.28 Affine chains An *affine k-chain* Γ in an open set $E \subset R^n$ is a collection of finitely many oriented affine k-simplexes $\sigma_1, \ldots, \sigma_r$ in E. These need not be distinct; a simplex may thus occur in Γ with a certain multiplicity.

If Γ is as above, and if ω is a k-form in E, we define

$$\int_\Gamma \omega = \sum_{i=1}^{r} \int_{\sigma_i} \omega. \tag{82}$$

We may view a k-surface Φ in E as a function whose domain is the collection of all k-forms in E and which assigns the number $\int_\Phi \omega$ to ω. Since real-valued functions can be added (as in Definition 4.3), this suggests the use of the notation

$$\Gamma = \sigma_1 + \cdots + \sigma_r \tag{83}$$

or, more compactly,

$$\Gamma = \sum_{i=1}^{r} \sigma_i \tag{84}$$

to state the fact that (82) holds for every k-form ω in E.

To avoid misunderstanding, we point out explicitly that the notations introduced by (83) and (80) have to be handled with care. The point is that every oriented affine k-simplex σ in R^n is a function in two ways, with different domains and different ranges, and that therefore two entirely different operations of addition are possible. Originally, σ was defined as an R^n-valued function with domain Q^k; accordingly, $\sigma_1 + \sigma_2$ *could* be interpreted to be the function σ that assigns the vector $\sigma_1(\mathbf{u}) + \sigma_2(\mathbf{u})$ to every $\mathbf{u} \in Q^k$; note that σ is then again an oriented affine k-simplex in R^n! This is *not* what is meant by (83).

For example, if $\sigma_2 = -\sigma_1$ as in (80) (that is to say, if σ_1 and σ_2 have the same set of vertices but are oppositely oriented) and if $\Gamma = \sigma_1 + \sigma_2$, then $\int_\Gamma \omega = 0$ for all ω, and we may express this by writing $\Gamma = 0$ or $\sigma_1 + \sigma_2 = 0$. This does not mean that $\sigma_1(\mathbf{u}) + \sigma_2(\mathbf{u})$ is the null vector of R^n.

10.29 Boundaries For $k \geq 1$, the *boundary* of the oriented affine k-simplex

$$\sigma = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_k]$$

is defined to be the affine $(k-1)$-chain

$$\partial\sigma = \sum_{j=0}^{k} (-1)^j [\mathbf{p}_0, \ldots, \mathbf{p}_{j-1}, \mathbf{p}_{j+1}, \ldots, \mathbf{p}_k]. \tag{85}$$

For example, if $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2]$, then

$$\partial\sigma = [\mathbf{p}_1, \mathbf{p}_2] - [\mathbf{p}_0, \mathbf{p}_2] + [\mathbf{p}_0, \mathbf{p}_1] = [\mathbf{p}_0, \mathbf{p}_1] + [\mathbf{p}_1, \mathbf{p}_2] + [\mathbf{p}_2, \mathbf{p}_0],$$

which coincides with the usual notion of the oriented boundary of a triangle.

For $1 \leq j \leq k$, observe that the simplex $\sigma_j = [\mathbf{p}_0, \ldots, \mathbf{p}_{j-1}, \mathbf{p}_{j+1}, \ldots, \mathbf{p}_k]$ which occurs in (85) has Q^{k-1} as its parameter domain and that it is defined by

$$\sigma_j(\mathbf{u}) = \mathbf{p}_0 + B\mathbf{u} \qquad (\mathbf{u} \in Q^{k-1}), \tag{86}$$

where B is the linear mapping from R^{k-1} to R^n determined by

$$B\mathbf{e}_i = \mathbf{p}_i - \mathbf{p}_0 \qquad (\text{if} \quad 1 \leq i \leq j-1),$$
$$B\mathbf{e}_i = \mathbf{p}_{i+1} - \mathbf{p}_0 \qquad (\text{if} \quad j \leq i \leq k-1).$$

The simplex

$$\sigma_0 = [\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k],$$

which also occurs in (85), is given by the mapping

$$\sigma_0(\mathbf{u}) = \mathbf{p}_1 + B\mathbf{u},$$

where $B\mathbf{e}_i = \mathbf{p}_{i+1} - \mathbf{p}_1$ for $1 \leq i \leq k-1$.

10.30 Differentiable simplexes and chains Let T be a $\mathscr{C}''$-mapping of an open set $E \subset R^n$ into an open set $V \subset R^m$; T need not be one-to-one. If σ is an oriented affine k-simplex in E, then the composite mapping $\Phi = T \circ \sigma$ (which we shall sometimes write in the simpler form $T\sigma$) is a k-surface in V, with parameter domain Q^k. We call Φ an *oriented k-simplex of class $\mathscr{C}''$*.

A finite collection Ψ of oriented k-simplexes $\Phi_1, \ldots, \Phi_r$ of class $\mathscr{C}''$ in V is called a *k-chain of class* $\mathscr{C}''$ in V. If ω is a k-form in V, we define

$$\int_\Psi \omega = \sum_{i=1}^{r} \int_{\Phi_i} \omega \tag{87}$$

and use the corresponding notation $\Psi = \Sigma \Phi_i$.

If $\Gamma = \Sigma \sigma_i$ is an affine chain and if $\Phi_i = T \circ \sigma_i$, we also write $\Psi = T \circ \Gamma$, or

$$T\left(\sum \sigma_i\right) = \sum T\sigma_i. \tag{88}$$

The boundary $\partial\Phi$ of the oriented k-simplex $\Phi = T \circ \sigma$ is defined to be the $(k-1)$ chain

$$\partial\Phi = T(\partial\sigma). \tag{89}$$

In justification of (89), observe that if T is affine, then $\Phi = T \circ \sigma$ is an oriented affine k-simplex, in which case (89) is not a matter of definition, but is seen to be a *consequence* of (85). Thus (89) generalizes this special case.

It is immediate that $\partial\Phi$ is of class $\mathscr{C}''$ if this is true of Φ.

Finally, we define the boundary $\partial\Psi$ of the k-chain $\Psi = \Sigma\Phi_i$ to be the $(k-1)$ chain

$$\partial\Psi = \sum \partial\Phi_i. \tag{90}$$

10.31 Positively oriented boundaries So far we have associated boundaries to chains, not to subsets of R^n. This notion of boundary is exactly the one that is most suitable for the statement and proof of Stokes' theorem. However, in applications, especially in R^2 or R^3, it is customary and convenient to talk about "oriented boundaries" of certain sets as well. We shall now describe this briefly.

Let Q^n be the standard simplex in R^n, let σ_0 be the identity mapping with domain Q^n. As we saw in Sec. 10.26, σ_0 may be regarded as a positively oriented n-simplex in R^n. Its boundary $\partial\sigma_0$ is an affine $(n-1)$-chain. This chain is called the *positively oriented boundary of the set* Q^n.

For example, the positively oriented boundary of Q^3 is

$$[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] - [\mathbf{0}, \mathbf{e}_2, \mathbf{e}_3] + [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_3] - [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2].$$

Now let T be a 1-1 mapping of Q^n into R^n, of class $\mathscr{C}''$, whose Jacobian is positive (at least in the interior of Q^n). Let $E = T(Q^n)$. By the inverse function theorem, E is the closure of an open subset of R^n. We define the positively oriented boundary of the set E to be the $(n-1)$-chain

$$\partial T = T(\partial\sigma_0),$$

and we may denote this $(n-1)$-chain by ∂E.

An obvious question occurs here: If $E = T_1(Q^n) = T_2(Q^n)$, and if both T_1 and T_2 have positive Jacobians, is it true that $\partial T_1 = \partial T_2$? That is to say, does the equality

$$\int_{\partial T_1} \omega = \int_{\partial T_2} \omega$$

hold for every $(n-1)$-form ω? The answer is yes, but we shall omit the proof. (To see an example, compare the end of this section with Exercise 17.)

One can go further. Let

$$\Omega = E_1 \cup \cdots \cup E_r,$$

where $E_i = T_i(Q^n)$, each T_i has the properties that T had above, and the interiors of the sets E_i are pairwise disjoint. Then the $(n-1)$-chain

$$\partial T_1 + \cdots + \partial T_r = \partial \Omega$$

is called the positively oriented boundary of Ω.

For example, the unit square I^2 in R^2 is the union of $\sigma_1(Q^2)$ and $\sigma_2(Q^2)$, where

$$\sigma_1(\mathbf{u}) = \mathbf{u}, \qquad \sigma_2(\mathbf{u}) = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{u}.$$

Both σ_1 and σ_2 have Jacobian $1 > 0$. Since

$$\sigma_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2], \qquad \sigma_2 = [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1]$$

we have

$$\partial\sigma_1 = [\mathbf{e}_1, \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1],$$

$$\partial\sigma_2 = [\mathbf{e}_2, \mathbf{e}_1] - [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2];$$

The sum of these two boundaries is

$$\partial I^2 = [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}],$$

the positively oriented boundary of I^2. Note that $[\mathbf{e}_1, \mathbf{e}_2]$ canceled $[\mathbf{e}_2, \mathbf{e}_1]$.

If Φ is a 2-surface in R^m, with parameter domain I^2, then Φ (regarded as a function on 2-forms) is the same as the 2-chain

$$\Phi \circ \sigma_1 + \Phi \circ \sigma_2.$$

Thus

$$\begin{aligned} \partial\Phi &= \partial(\Phi \circ \sigma_1) + \partial(\Phi \circ \sigma_2) \\ &= \Phi(\partial\sigma_1) + \Phi(\partial\sigma_2) = \Phi(\partial I^2). \end{aligned}$$

In other words, if the parameter domain of Φ is the square I^2, we need not refer back to the simplex Q^2, but can obtain $\partial\Phi$ directly from ∂I^2.

Other examples may be found in Exercises 17 to 19.

10.32 Example For $0 \le u \le \pi$, $0 \le v \le 2\pi$, define

$$\Sigma(u, v) = (\sin u \cos v, \sin u \sin v, \cos u).$$

Then Σ is a 2-surface in R^3, whose parameter domain is a rectangle $D \subset R^2$, and whose range is the unit sphere in R^3. Its boundary is

$$\partial\Sigma = \Sigma(\partial D) = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$$

where

$$\begin{aligned} \gamma_1(u) &= \Sigma(u, 0) = (\sin u, 0, \cos u), \\ \gamma_2(v) &= \Sigma(\pi, v) = (0, 0, -1), \\ \gamma_3(u) &= \Sigma(\pi - u, 2\pi) = (\sin u, 0, -\cos u), \\ \gamma_4(v) &= \Sigma(0, 2\pi - v) = (0, 0, 1), \end{aligned}$$

with $[0, \pi]$ and $[0, 2\pi]$ as parameter intervals for u and v, respectively.

Since γ_2 and γ_4 are constant, their derivatives are 0, hence the integral of any 1-form over γ_2 or γ_4 is 0. [See Example 1.12(*a*).]

Since $\gamma_3(u) = \gamma_1(\pi - u)$, direct application of (35) shows that

$$\int_{\gamma_3} \omega = -\int_{\gamma_1} \omega$$

for every 1-form ω. Thus $\int_{\partial\Sigma} \omega = 0$, and we conclude that $\partial\Sigma = 0$.

(In geographic terminology, $\partial\Sigma$ starts at the north pole N, runs to the south pole S along a meridian, pauses at S, returns to N along the same meridian, and finally pauses at N. The two passages along the meridian are in opposite directions. The corresponding two line integrals therefore cancel each other. In Exercise 32 there is also one curve which occurs twice in the boundary, but without cancellation.)

STOKES' THEOREM

10.33 Theorem *If Ψ is a k-chain of class $\mathscr{C}''$ in an open set $V \subset R^m$ and if ω is a $(k-1)$-form of class $\mathscr{C}'$ in V, then*

$$\int_{\Psi} d\omega = \int_{\partial\Psi} \omega. \tag{91}$$

The case $k = m = 1$ is nothing but the fundamental theorem of calculus (with an additional differentiability assumption). The case $k = m = 2$ is Green's theorem, and $k = m = 3$ gives the so-called "divergence theorem" of Gauss. The case $k = 2$, $m = 3$ is the one originally discovered by Stokes. (Spivak's book describes some of the historical background.) These special cases will be discussed further at the end of the present chapter.

Proof It is enough to prove that

$$\int_{\Phi} d\omega = \int_{\partial\Phi} \omega \tag{92}$$

for every oriented k-simplex Φ of class $\mathscr{C}''$ in V. For if (92) is proved and if $\Psi = \Sigma\Phi_i$, then (87) and (89) imply (91).

Fix such a Φ and put

$$\sigma = [\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_k]. \tag{93}$$

Thus σ is the oriented affine k-simplex with parameter domain Q^k which is defined by the identity mapping. Since Φ is also defined on Q^k (see Definition 10.30) and $\Phi \in \mathscr{C}''$, there is an open set $E \subset R^k$ which contains Q^k, and there is a $\mathscr{C}''$-mapping T of E into V such that $\Phi = T \circ \sigma$. By Theorems 10.25 and 10.22(c), the left side of (92) is equal to

$$\int_{T\sigma} d\omega = \int_{\sigma} (d\omega)_T = \int_{\sigma} d(\omega_T).$$

Another application of Theorem 10.25 shows, by (89), that the right side of (92) is

$$\int_{\partial(T\sigma)} \omega = \int_{T(\partial\sigma)} \omega = \int_{\partial\sigma} \omega_T.$$

Since ω_T is a $(k-1)$-form in E, we see that *in order to prove* (92) *we merely have to show that*

$$\int_{\sigma} d\lambda = \int_{\partial\sigma} \lambda \tag{94}$$

for the special simplex (93) *and for every* $(k-1)$*-form* λ *of class* $\mathscr{C}'$ *in* E.

If $k = 1$, the definition of an oriented 0-simplex shows that (94) merely asserts that

$$\int_0^1 f'(u)\, du = f(1) - f(0) \tag{95}$$

for every continuously differentiable function f on $[0, 1]$, which is true by the fundamental theorem of calculus.

From now on we assume that $k > 1$, fix an integer r $(1 \le r \le k)$, and choose $f \in \mathscr{C}'(E)$. It is then enough to prove (94) for the case

$$\lambda = f(\mathbf{x})\, dx_1 \wedge \cdots \wedge dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_k \tag{96}$$

since every $(k-1)$-form is a sum of these special ones, for $r = 1, \ldots, k$.

By (85), the boundary of the simplex (93) is

$$\partial\sigma = [\mathbf{e}_1, \ldots, \mathbf{e}_k] + \sum_{i=1}^{k} (-1)^i \tau_i$$

where

$$\tau_i = [\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \ldots, \mathbf{e}_k]$$

for $i = 1, \ldots, k$. Put

$$\tau_0 = [\mathbf{e}_r, \mathbf{e}_1, \ldots, \mathbf{e}_{r-1}, \mathbf{e}_{r+1}, \ldots, \mathbf{e}_k].$$

Note that τ_0 is obtained from $[\mathbf{e}_1, \ldots, \mathbf{e}_k]$ by $r - 1$ successive interchanges of $\mathbf{e}_r$ and its left neighbors. Thus

(97)
$$\partial\sigma = (-1)^{r-1}\tau_0 + \sum_{i=1}^{k} (-1)^i \tau_i .$$

Each τ_i has Q^{k-1} as parameter domain.

If $\mathbf{x} = \tau_0(\mathbf{u})$ and $\mathbf{u} \in Q^{k-1}$, then

(98)
$$x_j = \begin{cases} u_j & (1 \le j < r), \\ 1 - (u_1 + \cdots + u_{k-1}) & (j = r), \\ u_{j-1} & (r < j \le k). \end{cases}$$

If $1 \le i \le k$, $\mathbf{u} \in Q^{k-1}$, and $\mathbf{x} = \tau_i(\mathbf{u})$, then

(99)
$$x_j = \begin{cases} u_j & (1 \le j < i), \\ 0 & (j = i), \\ u_{j-1} & (i < j \le k). \end{cases}$$

For $0 \le i \le k$, let J_i be the Jacobian of the mapping

(100)
$$(u_1, \ldots, u_{k-1}) \to (x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_k)$$

induced by τ_i. When $i = 0$ and when $i = r$, (98) and (99) show that (100) is the identity mapping. Thus $J_0 = 1$, $J_r = 1$. For other i, the fact that $x_i = 0$ in (99) shows that J_i has a row of zeros, hence $J_i = 0$. Thus

(101)
$$\int_{\tau_i} \lambda = 0 \qquad (i \ne 0, i \ne r),$$

by (35) and (96). Consequently, (97) gives

(102)
$$\int_{\partial\sigma} \lambda = (-1)^{r-1} \int_{\tau_0} \lambda + (-1)^r \int_{\tau_r} \lambda$$
$$= (-1)^{r-1} \int \qquad [f(\tau_0(\mathbf{u})) - f(\tau_r(\mathbf{u}))]\, d\mathbf{u}.$$

On the other hand,

$$d\lambda = (D_r f)(\mathbf{x})dx_r \wedge dx_1 \wedge \cdots \wedge dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_k$$
$$= (-1)^{r-1}(D_r f)(\mathbf{x})\, dx_1 \wedge \cdots \wedge dx_k$$

so that

(103) $$\int_\sigma d\lambda = (-1)^{r-1} \int_{Q^k} (D_r f)(\mathbf{x})\, d\mathbf{x}.$$

We evaluate (103) by first integrating with respect to x_r, over the interval

$$[0, 1 - (x_1 + \cdots + x_{r-1} + x_{r+1} + \cdots + x_k)],$$

put $(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_k) = (u_1, \ldots, u_{k-1})$, and see with the aid of (98) that the integral over Q^k in (103) is equal to the integral over Q^{k-1} in (102). Thus (94) holds, and the proof is complete.

CLOSED FORMS AND EXACT FORMS

10.34 Definition Let ω be a k-form in an open set $E \subset R^n$. If there is a $(k-1)$-form λ in E such that $\omega = d\lambda$, then ω is said to be *exact in E*.

If ω is of class $\mathscr{C}'$ and $d\omega = 0$, then ω is said to be *closed*.

Theorem 10.20(*b*) shows that every exact form of class $\mathscr{C}'$ is closed.

In certain sets E, for example in convex ones, the converse is true; this is the content of Theorem 10.39 (usually known as *Poincaré's lemma*) and Theorem 10.40. However, Examples 10.36 and 10.37 will exhibit closed forms that are not exact.

10.35 Remarks

(*a*) Whether a given k-form ω is or is not closed can be verified by simply differentiating the coefficients in the standard presentation of ω. For example, a 1-form

(104) $$\omega = \sum_{i=1}^{n} f_i(\mathbf{x})\, dx_i,$$

with $f_i \in \mathscr{C}'(E)$ for some open set $E \subset R^n$, is closed if and only if the equations

(105) $$(D_j f_i)(\mathbf{x}) = (D_i f_j)(\mathbf{x})$$

hold for all i, j in $\{1, \ldots, n\}$ and for all $\mathbf{x} \in E$.

Note that (105) is a "pointwise" condition; it does not involve any global properties that depend on the shape of E.

On the other hand, to show that ω is exact in E, one has to prove the existence of a form λ, defined in E, such that $d\lambda = \omega$. This amounts to solving a system of partial differential equations, not just locally, but in all of E. For example, to show that (104) is exact in a set E, one has to find a function (or 0-form) $g \in \mathscr{C}'(E)$ such that

$$(D_i g)(\mathbf{x}) = f_i(\mathbf{x}) \qquad (\mathbf{x} \in E, 1 \leq i \leq n). \tag{106}$$

Of course, (105) is a necessary condition for the solvability of (106).

(*b*) Let ω be an *exact* k-form in E. Then there is a $(k-1)$-form λ in E with $d\lambda = \omega$, and Stokes' theorem asserts that

$$\int_\Psi \omega = \int_\Psi d\lambda = \int_{\partial\Psi} \lambda \tag{107}$$

for every k-chain Ψ of class $\mathscr{C}''$ in E.

If Ψ_1 and Ψ_2 are such chains, and if they have the same boundaries, it follows that

$$\int_{\Psi_1} \omega = \int_{\Psi_2} \omega.$$

In particular, *the integral of an exact k-form in E is* 0 *over every k-chain in E whose boundary is* 0.

As an important special case of this, note that integrals of exact 1-forms in E are 0 over closed (differentiable) curves in E.

(*c*) Let ω be a *closed* k-form in E. Then $d\omega = 0$, and Stokes' theorem asserts that

$$\int_{\partial\Psi} \omega = \int_\Psi d\omega = 0 \tag{108}$$

for every $(k+1)$-chain Ψ of class $\mathscr{C}''$ in E.

In other words, *integrals of closed k-forms in E are* 0 *over k-chains that are boundaries of $(k+1)$-chains in E.*

(*d*) Let Ψ be a $(k+1)$-chain in E and let λ be a $(k-1)$-form in E, both of class $\mathscr{C}''$. Since $d^2\lambda = 0$, two applications of Stokes' theorem show that

$$\int_{\partial\partial\Psi} \lambda = \int_{\partial\Psi} d\lambda = \int_\Psi d^2\lambda = 0. \tag{109}$$

We conclude that $\partial^2\Psi = 0$. In other words, *the boundary of a boundary is* 0.

See Exercise 16 for a more direct proof of this.

10.36 Example Let $E = R^2 - \{\mathbf{0}\}$, the plane with the origin removed. The 1-form

$$\eta = \frac{x\,dy - y\,dx}{x^2 + y^2} \tag{110}$$

is *closed* in $R^2 - \{\mathbf{0}\}$. This is easily verified by differentiation. Fix $r > 0$, and define

$$\gamma(t) = (r \cos t, r \sin t) \qquad (0 \le t \le 2\pi). \tag{111}$$

Then γ is a curve (an "oriented 1-simplex") in $R^2 - \{\mathbf{0}\}$. Since $\gamma(0) = \gamma(2\pi)$, we have

$$\partial\gamma = 0. \tag{112}$$

Direct computation shows that

$$\int_\gamma \eta = 2\pi \ne 0. \tag{113}$$

The discussion in Remarks 10.35(*b*) and (*c*) shows that we can draw two conclusions from (113):

First, *η is not exact in $R^2 - \{\mathbf{0}\}$*, for otherwise (112) would force the integral (113) to be 0.

Secondly, *γ is not the boundary of any 2-chain in $R^2 - \{\mathbf{0}\}$* (of class $\mathscr{C}''$), for otherwise the fact that η is closed would force the integral (113) to be 0.

10.37 Example Let $E = R^3 - \{\mathbf{0}\}$, 3-space with the origin removed. Define

$$\zeta = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}} \tag{114}$$

where we have written (x, y, z) in place of (x_1, x_2, x_3). Differentiation shows that $d\zeta = 0$, so that ζ is a closed 2-form in $R^3 - \{\mathbf{0}\}$.

Let Σ be the 2-chain in $R^3 - \{\mathbf{0}\}$ that was constructed in Example 10.32; recall that Σ is a parametrization of the unit sphere in R^3. Using the rectangle D of Example 10.32 as parameter domain, it is easy to compute that

$$\int_\Sigma \zeta = \int_D \sin u\,du\,dv = 4\pi \ne 0. \tag{115}$$

As in the preceding example, we can now conclude that ζ is *not* exact in $R^3 - \{\mathbf{0}\}$ (since $\partial\Sigma = 0$, as was shown in Example 10.32) and that the sphere Σ is not the boundary of any 3-chain in $R^3 - \{\mathbf{0}\}$ (of class $\mathscr{C}''$), although $\partial\Sigma = 0$.

The following result will be used in the proof of Theorem 10.39.

10.38 Theorem *Suppose E is a convex open set in R^n, $f \in \mathscr{C}'(E)$, p is an integer, $1 \le p \le n$, and*

$$(D_j f)(\mathbf{x}) = 0 \qquad (p < j \le n, \mathbf{x} \in E). \tag{116}$$

Then there exists an $F \in \mathscr{C}'(E)$ such that

$$(D_p F)(\mathbf{x}) = f(\mathbf{x}), \qquad (D_j F)(\mathbf{x}) = 0 \qquad (p < j \le n, \mathbf{x} \in E). \tag{117}$$

Proof Write $\mathbf{x} = (\mathbf{x}', x_p, \mathbf{x}'')$, where

$$\mathbf{x}' = (x_1, \ldots, x_{p-1}),\ \mathbf{x}'' = (x_{p+1}, \ldots, x_n).$$

(When $p = 1$, $\mathbf{x}'$ is absent; when $p = n$, $\mathbf{x}''$ is absent.) Let V be the set of all $(\mathbf{x}', x_p) \in R^p$ such that $(\mathbf{x}', x_p, \mathbf{x}'') \in E$ for some $\mathbf{x}''$. Being a projection of E, V is a convex open set in R^p. Since E is convex and (116) holds, $f(\mathbf{x})$ does not depend on $\mathbf{x}''$. Hence there is a function φ, with domain V, such that

$$f(\mathbf{x}) = \varphi(\mathbf{x}', x_p)$$

for all $\mathbf{x} \in E$.

If $p = 1$, V is a segment in R^1 (possibly unbounded). Pick $c \in V$ and define

$$F(\mathbf{x}) = \int_c^{x_1} \varphi(t)\, dt \qquad (\mathbf{x} \in E).$$

If $p > 1$, let U be the set of all $\mathbf{x}' \in R^{p-1}$ such that $(\mathbf{x}', x_p) \in V$ for some x_p. Then U is a convex open set in R^{p-1}, and there is a function $\alpha \in \mathscr{C}'(U)$ such that $(\mathbf{x}', \alpha(\mathbf{x}')) \in V$ for every $\mathbf{x}' \in U$; in other words, the graph of α lies in V (Exercise 29). Define

$$F(\mathbf{x}) = \int_{\alpha(\mathbf{x}')}^{x_p} \varphi(\mathbf{x}', t)\, dt \qquad (\mathbf{x} \in E).$$

In either case, F satisfies (117).

(*Note:* Recall the usual convention that $\int_a^b$ means $-\int_b^a$ if $b < a$.)

10.39 Theorem *If $E \subset R^n$ is convex and open, if $k \ge 1$, if ω is a k-form of class $\mathscr{C}'$ in E, and if $d\omega = 0$, then there is a $(k-1)$-form λ in E such that $\omega = d\lambda$.*

Briefly, closed forms are exact in convex sets.

Proof For $p = 1, \ldots, n$, let Y_p denote the set of all k-forms ω, of class $\mathscr{C}'$ in E, whose standard presentation

$$\omega = \sum_I f_I(\mathbf{x})\, dx_I \tag{118}$$

does not involve $dx_{p+1}, \ldots, dx_n$. In other words, $I \subset \{1, \ldots, p\}$ if $f_I(\mathbf{x}) \ne 0$ for some $\mathbf{x} \in E$.

We shall proceed by induction on p.

Assume first that $\omega \in Y_1$. Then $\omega = f(\mathbf{x})\,dx_1$. Since $d\omega = 0$, $(D_j f)(\mathbf{x}) = 0$ for $1 < j \le n$, $\mathbf{x} \in E$. By Theorem 10.38 there is an $F \in \mathscr{C}'(E)$ such that $D_1 F = f$ and $D_j F = 0$ for $1 < j \le n$. Thus

$$dF = (D_1 F)(\mathbf{x})\,dx_1 = f(\mathbf{x})\,dx_1 = \omega.$$

Now we take $p > 1$ and make the following induction hypothesis: *Every closed k-form that belongs to Y_{p-1} is exact in E.*

Choose $\omega \in Y_p$ so that $d\omega = 0$. By (118),

$$\sum_I \sum_{j=1}^{n} (D_j f_I)(\mathbf{x})\,dx_j \wedge dx_I = d\omega = 0. \tag{119}$$

Consider a fixed j, with $p < j \le n$. Each I that occurs in (118) lies in $\{1, \ldots, p\}$. If I_1, I_2 are two of these k-indices, and if $I_1 \ne I_2$, then the $(k + 1)$-indices (I_1, j), (I_2, j) are distinct. Thus there is no cancellation, and we conclude from (119) that every coefficient in (118) satisfies

$$(D_j f_I)(\mathbf{x}) = 0 \qquad (\mathbf{x} \in E, p < j \le n). \tag{120}$$

We now gather those terms in (118) that contain dx_p and rewrite ω in the form

$$\omega = \alpha + \sum_{I_0} f_I(\mathbf{x})\,dx_{I_0} \wedge dx_p, \tag{121}$$

where $\alpha \in Y_{p-1}$, each I_0 is an increasing $(k-1)$-index in $\{1, \ldots, p-1\}$, and $I = (I_0, p)$. By (120), Theorem 10.38 furnishes functions $F_I \in \mathscr{C}'(E)$ such that

$$D_p F_I = f_I, \qquad D_j F_I = 0 \qquad (p < j \le n). \tag{122}$$

Put

$$\beta = \sum_{I_0} F_I(\mathbf{x})\,dx_{I_0} \tag{123}$$

and define $\gamma = \omega - (-1)^{k-1}\,d\beta$. Since β is a $(k-1)$-form, it follows that

$$\begin{aligned}\gamma &= \omega - \sum_{I_0} \sum_{j=1}^{p} (D_j F_I)(\mathbf{x})\,dx_{I_0} \wedge dx_j \\ &= \alpha - \sum_{I_0} \sum_{j=1}^{p-1} (D_j F_I)(\mathbf{x})\,dx_{I_0} \wedge dx_j,\end{aligned}$$

which is clearly in Y_{p-1}. Since $d\omega = 0$ and $d^2\beta = 0$, we have $d\gamma = 0$. Our induction hypothesis shows therefore that $\gamma = d\mu$ for some $(k-1)$-form μ in E. If $\lambda = \mu + (-1)^{k-1}\beta$, we conclude that $\omega = d\lambda$.

By induction, this completes the proof.

10.40 Theorem *Fix k, $1 \le k \le n$. Let $E \subset R^n$ be an open set in which every closed k-form is exact. Let T be a 1-1 $\mathscr{C}''$-mapping of E onto an open set $U \subset R^n$ whose inverse S is also of class $\mathscr{C}''$.*

Then every closed k-form in U is exact in U.

Note that every convex open set E satisfies the present hypothesis, by Theorem 10.39. The relation between E and U may be expressed by saying that they are *$\mathscr{C}''$-equivalent.*

Thus every closed form is exact in any set which is $\mathscr{C}''$-equivalent to a convex open set.

Proof Let ω be a k-form in U, with $d\omega = 0$. By Theorem 10.22(c), ω_T is a k-form in E for which $d(\omega_T) = 0$. Hence $\omega_T = d\lambda$ for some $(k-1)$-form λ in E. By Theorem 10.23, and another application of Theorem 10.22(c),

$$\omega = (\omega_T)_S = (d\lambda)_S = d(\lambda_S).$$

Since λ_S is a $(k-1)$-form in U, ω is exact in U.

10.41 Remark In applications, cells (see Definition 2.17) are often more convenient parameter domains than simplexes. If our whole development had been based on cells rather than simplexes, the computation that occurs in the proof of Stokes' theorem would be even simpler. (It is done that way in Spivak's book.) The reason for preferring simplexes is that the definition of the boundary of an oriented simplex seems easier and more natural than is the case for a cell. (See Exercise 19.) Also, the partitioning of sets into simplexes (called "triangulation") plays an important role in topology, and there are strong connections between certain aspects of topology, on the one hand, and differential forms, on the other. These are hinted at in Sec. 10.35. The book by Singer and Thorpe contains a good introduction to this topic.

Since every cell can be triangulated, we may regard it as a chain. For dimension 2, this was done in Example 10.32; for dimension 3, see Exercise 18.

Poincaré's lemma (Theorem 10.39) can be proved in several ways. See, for example, page 94 in Spivak's book, or page 280 in Fleming's. Two simple proofs for certain special cases are indicated in Exercises 24 and 27.

VECTOR ANALYSIS

We conclude this chapter with a few applications of the preceding material to theorems concerning vector analysis in R^3. These are special cases of theorems about differential forms, but are usually stated in different terminology. We are thus faced with the job of translating from one language to another.

page 123, lines −15 to −14

Verification that the permutation $\tau_{p,q}$

$$\begin{matrix} 1 \dots p & p+1 \dots p+q \\ & \downarrow \\ p+1 \dots p+q & 1 \dots p \end{matrix}$$

has sign $(-1)^{pq}$.

Follow the suggestion on line −14, fixing p and proceeding by induction on q.

$q=1$ The permutation is just the cycle $(1 \dots p \; p+1)$, which has sign $(-1)^p$.

Assume for $q=k$, and suppose $q=k+1$.

Then $\operatorname{sgn} \tau_{p,k} = (-1)^{pk}$ and $\tau_{p,k}$ is

$$\begin{matrix} 1 \dots p & p+1 \dots p+k & p+k+1 \\ & \downarrow & \\ p+1 \dots p+k & 1 \dots p & p+k+1 \end{matrix}$$

The latter implies $\tau_{p,k+1} = (k+1 \dots p+k+1) \circ \tau_{p,k}$ and hence has sign $= (-1)^p \cdot (-1)^{pk} = (-1)^{p(k+1)}$ completing the verification of the inductive step.

page 123, lines -11 to -10

Verification of Proposition 4.

By construction there is a commutative diagram (Ω = quotient projection)

$$\begin{array}{ccc}
Cov_p(U) \times Cov_q(U) & \xrightarrow{\otimes} & Cov_{p+q}(U) \\
\downarrow \Omega \times \Omega & & \downarrow \Omega \\
\Lambda^p(U) \times \Lambda^q(U) & \xrightarrow{\wedge} & \Lambda^{p+q}(U)
\end{array}$$

for all p and q, so if $\overline{\lambda}, \overline{\omega}, \overline{\theta} \in Cov_*(U)$ project to λ, ω, θ we have

$$(\theta \wedge \omega) \wedge \lambda = (\Omega\overline{\theta} \wedge \Omega\overline{\omega}) \wedge \Omega\overline{\lambda} =$$

$$\Omega(\overline{\theta} \otimes \overline{\omega}) \wedge \Omega(\overline{\lambda}) = \Omega((\overline{\theta} \otimes \overline{\omega}) \otimes \overline{\lambda}) \;\; \underset{\text{associative}}{\overset{\text{since } \otimes \text{ is}}{=\!=\!=}}$$

$$\Omega(\overline{\theta} \otimes (\overline{\omega} \otimes \overline{\lambda})) = \Omega(\overline{\theta}) \wedge \Omega(\overline{\omega} \otimes \overline{\lambda}) =$$

$$\Omega(\overline{\theta}) \wedge (\Omega\overline{\omega} \wedge \Omega\overline{\lambda}) = \theta \wedge (\omega \wedge \lambda).$$

page ~~124~~ 125, lines 1-6

Proof of Theorem 5

(i) Since both the left and right hand sides are $\mathbb{R}$-bilinear with respect to θ and ω, it suffices to verify this result for sets of forms θ_α and ω_α which span $\Lambda^p(U)$ and $\Lambda^q(U)$. The obvious choices are forms given by $f\,dx^{i_1}\wedge\ldots\wedge dx^{i_p}$ and $g\,dx^{j_1}\wedge\ldots\wedge dx^{j_q}$ where $f, g \in C^\infty(U)$.

PRELIMINARY Note

The relationship

$$\text{exterior derivative of coord. fcn } x^i = \text{"}dx^i\text{"}$$

looks tautological, but it isn't quite. One has $d(x^i) = \sum \frac{\partial x^i}{\partial x^j}\,\text{"}dx^j\text{"}$ which reduces to "dx^i" because $\frac{\partial x^i}{\partial x^j} = \delta_{ij}$ (Kronecker delta).

<u>page 124, lines 1-6 continued</u>

So we need to compare

$$d((f\,dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge(g\,dx^{j_1}\wedge\cdots\wedge dx^{j_q}))$$ and

$$d(f\,dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge(g\,dx^{j_1}\wedge\cdots\wedge dx^{j_q}) +$$
$$(-1)^p(f\,dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge d(g\,dx^{j_1}\wedge\cdots\wedge dx^{j_q}).$$

The first of these is

$$d(fg)\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q} =$$
$$\sum f\frac{\partial g}{\partial x^k}dx^k\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q} +$$
$$\sum\frac{\partial f}{\partial x^l}g\,dx^l\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q} =$$
$$(-1)^p(f\,dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge(dg\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q}) +$$
$$(df\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge(g\,dx^{j_1}\wedge\cdots\wedge dx^{j_q}) =$$
$$(-1)^p\,\theta\wedge d\omega + d\theta\wedge\omega.$$

page 124, lines 1-6 continued

(ii) Since $d \circ d$ is $\mathbb{R}$-linear, as in (i) it suffices to prove it for a form of type $f\,dx^{i_1} \wedge \cdots \wedge dx^{i_p}$.

It is routine to check that $ddx^i = 0$ for all i, and hence by (i) we have $d(dx^{i_1} \wedge \cdots \wedge dx^{i_p}) = 0$.

Now $df = \sum_k \frac{\partial f}{\partial x^k}\, dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ and

$$ddf = \sum_{j,k} \frac{\partial^2 f}{\partial x^j \partial x^k}\, dx^j \wedge dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} =$$

$$\sum_{j<k} + \sum_{j=k} + \sum_{j>k}.$$ The middle term vanishes because $dx^j \wedge dx^j = 0$, and the first and third cancel because

$$\frac{\partial^2 f}{\partial x^j \partial x^k}\, dx^j \wedge dx^k = -\frac{\partial^2 f}{\partial x^k \partial x^j}\, dx^k \wedge dx^j \quad \text{if } j \neq k.$$

page 125 lines 10-12

(i) In this case we have an isomorphism of free $C^\infty(U)$ modules

$$\Phi_1 : \text{Vector Fields} \longrightarrow \text{1-forms}$$

with Φ_1 of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ equal to dx, dy, dz.

Then $\nabla f = \frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial}{\partial z}$, and

$$\Phi_1(\nabla f) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

(ii) In this case Φ_2 sends $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ to $dy \wedge dz$, $dz \wedge dx$ [NOTE!!], $dx \wedge dy$ and

$$d\,\Phi_1\left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}\right) = d(P\,dx + Q\,dy + R\,dz)$$

$$= \frac{\partial P}{\partial y} dy\wedge dx + \frac{\partial P}{\partial z} dz\wedge dx + \frac{\partial Q}{\partial x} dx\wedge dy + \frac{\partial Q}{\partial z} dz\wedge dy + \frac{\partial R}{\partial x} dx\wedge dz + \frac{\partial R}{\partial y} dy\wedge dz = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\wedge dy$$

<u>page 125, lines 10-12 continued</u>

and the latter is equal to

$$\Phi_2\Big(\underset{\text{CURL}}{\nabla\times}\big(P\tfrac{\partial}{\partial x}+Q\tfrac{\partial}{\partial y}+R\tfrac{\partial}{\partial z}\big)\Big)$$

(iii) $d\,\Phi_2\big(P\frac{\partial}{\partial x}+Q\frac{\partial}{\partial y}+R\frac{\partial}{\partial z}\big)=$

$d\,(P\,dy\wedge dz+Q\,dz\wedge dx+R\,dx\wedge dy)=$

$\frac{\partial P}{\partial x}\,dx\wedge dy\wedge dz+\frac{\partial Q}{\partial y}\,dy\wedge dz\wedge dx+\frac{\partial R}{\partial z}\,dz\wedge dx\wedge dy$

$$=\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\wedge dy\wedge dz=$$

$$\Phi_3\Big(\underset{\text{DIV}}{\nabla\cdot}\big(P\tfrac{\partial}{\partial x}+Q\tfrac{\partial}{\partial y}+R\tfrac{\partial}{\partial z}\big)\Big).$$

page 138, lines 5-7

The decomposition $U = \cup K_m$, where

(i) $U \subseteq \mathbb{R}^n$ is open

(ii) K_m is compact and $K_m \subseteq \text{Int}\, K_{m+1}$

is actually a special case of the following purely point-set theoretic result.

Proposition Let X be locally compact T_2 and second countable. Then $X = \cup K_m$, where K_m is compact and $K_m \subseteq \text{Int}\, K_{m+1}$.

Proof ① Claim X has a countable base of open sets U_k such that $\overline{U_k}$ is compact.

Let $\mathfrak{V} = \{V_p\}_{p \geq 0}$ be a base, and let $x \in X$, so $x \in V_{p_0}$ some p_0. Then there is an open set W s.t. $x \in W \subseteq \overline{W} \subseteq V_{p_0}$ and $\overline{W}$ is compact. Since $\mathfrak{V}$ is a base for X, we know that

$W = \bigcup_{j \in J} V_j$ for some $J \subseteq \mathbb{N}$, and each $\overline{V_j}$ is compact since $\overline{V_j} \subseteq \overline{W}$ and the latter is compact. Hence the set $\mathfrak{V}'$ of all $V_q \in \mathfrak{V}$ with $\overline{V_q}$ compact is also a base for X.

(2) By the preceding, there is a countable base $\mathcal{U} = \{U_\ell\}_{\ell \geq 0}$ with compact closures. Pick $x_0 \in X$, and let $x_0 \in U_{k_0}$, and set $K_0 = \overline{U_{k_0} \cup U_0}$. Given K_m, construct K_{m+1} as follows: One can find a finite collection of open sets $\{U_j\}_{j \in J \subseteq \mathbb{N}}$ in $\mathcal{U}$ such that $K_m \subseteq \bigcup_{j \in J} U_j$. Let K_{m+1} be the closure of $\left(\bigcup_{j \in J} U_j\right) \cup U_{m+1}$. Clearly $K_m \subseteq \operatorname{Int} K_{m+1}$, K_{m+1} is also compact, and $X = \bigcup_{m \in \mathbb{N}} U_m \subseteq \bigcup K_m \subseteq X$ shows $X = \bigcup_m K_m$.

page 138, lines -12 to -5

Direct products of cochain complexes

Let $\{(C_\alpha, \delta_\alpha)\}$ be an indexed family of cochain complexes. Then $(\prod C_\alpha, \prod \delta_\alpha)$ is a chain complex because $(\prod \delta_\alpha)^2 \overset{\text{functoriality of direct product}}{=}$

$$\prod \delta_\alpha^2 = \prod 0 = 0.$$

Claim The projection maps

$\prod_\alpha H^*(C_\alpha) \to H^*(C_\gamma)$ induce an isomorphism
some γ

$$H^*\left(\prod_\alpha C_\alpha\right) \to \prod_\alpha H^*(C_\alpha).$$

One needs to check that $(x_\alpha) \in \prod C_\alpha$ is a cocycle/coboundary $\Leftrightarrow$ each x_γ is,
and then one can use the fact that if A_α is a submodule of B_α for all α, then

$$\prod_\alpha (A_\alpha / B_\alpha) \cong \left[\prod_\alpha A_\alpha\right] / \left[\prod_\alpha B_\alpha\right].$$