Note on the Hilbert Cube — II

This document may be viewed as a footnote to

`http://math.ucr.edu/∼res/math205A-2014/hilbert-cube.pdf`

and its purpose is to prove the following result:

PROPOSITION. *Define the Hilbert cube* **HQ** *to be a cartesian product of* $\aleph_0$ *copies of the unit interval* $[0,1]$. *Then there is no positive integer* n *such that* **HQ** *is homeomorphic to a subset of* $\mathbb{R}^n$.

Proof. Suppose that $X \subset \mathbb{R}^n$ is homeomorphic to **HQ** for some n, and consider the subset $B_{n+1} \subset \mathbf{HQ}$ consisting of all points $(x_1, \cdots)$ such that $x_k = 0$ if $k > n+1$. If $h : \mathbf{HQ} \to \mathbb{R}^n$ is given by the homeomorphism $\mathbf{HQ} \cong X$, let f be the restriction of h to the set V of points $(x_1, \cdots)$ such that $x_k = 0$ if $k > n+1$ and $0 < x_k < 1$ if $k \leq n+1$. Then f determines a continuous 1–1 map from V, which is homeomorphic to an open subset in $\mathbb{R}^{n+1}$, into $\mathbb{R}^n$, and if $j : \mathbb{R}^n \to \mathbb{R}^{n+1}$ sends v to $(v, 0)$ then f also determines a continuous 1–1 map $j \circ f$ from V into $\mathbb{R}^{n+1}$. By Invariance of Domain the mapping $j \circ f$ is open; however, by construction the image of $j \circ f$ is **NOT** open in $\mathbb{R}^{n+1}$. The source of the contradiction was our assumption about the existence of a continuous 1–1 mapping from **HQ** to $\mathbb{R}^n$, and therefore no such mapping can exist.∎

Note that if ℓ_2 is the infinite-dimensional Hilbert space consisting of all real valued sequences $(x_1, \cdots)$ such that

$$\sum_{i=1}^{\infty} x_i^2 \quad < \quad \infty$$

then there is a 1–1 continuous mapping from **HQ** to ℓ_2 sending $(x_1, \cdots) \in \mathbf{HQ}$ to

$$(x_1, \cdots, x_k/k, \cdots) \quad \in \quad \ell_2$$

and since **HQ** is compact (see the file cited above) it follows that this map defines a homeomorphism from **HQ** onto its image.