Homology and cell attachment

(Insertion for page 27 of `advancednotes2012.pdf`)

We now have the following natural question: *If X is obtained from the compact Hausdorff space A be adjoining a finite number — say M — of k-cells, how are the homology groups of X and A related?*

Since relative homology groups are the key to understanding such relationships, we proceed directly to the main result on such groups.

THEOREM A. *Suppose that X is obtained from the compact Hausdorff space A be adjoining a finite number M of k-cells. Then the homology groups $H_i(X,A)$ are trivial if $i \neq k$, and $H_k(X,A)$ is free abelian on M generators. In fact, if the cell attaching maps are $f_j : (D^k, S^{k-1}) \to (X,A)$ and $F : \coprod^M (D^k, S^{k-1}) \to (X,A)$ is the map whose restriction to the $j^{\rm th}$ disk in the domain is equal to f_i, then the map*

$$F_* : H_* \left(\textstyle\coprod_j (D^k, S^{k-1}) \right) \;\cong\; \oplus_j \; H_*(D^k, S^{k-1}) \;\longrightarrow\; H_*(X,A)$$

is an isomorphism.

Before proving this result we mention one important corollary:

COROLLARY B. *In the setting of the theorem, the map induced by inclusion from $H_*(A)$ to $H_*(X)$ is an isomorphism if $i \neq k, k-1$.*

The corollary follows from the fact that the relative homology groups are zero in all dimensions except k and the long exact homology sequence of the pair (X,A).■

Proof of Theorem A. The argument is similar to the proof on pages 104–105 of `algtop-notes2012.pdf`.

Let W be an open neighborhood of A in X given by the construction in Proposition 4, so that

$$A \;\subset\; W \subset \; \overline{W} \;\; \subset X$$

and A is a strong deformation retract of both W and $\overline{W}$. By construction we can choose W such that $X - W$ is the union of the sets $f_j[\frac{1}{2}D^k]$, where $r\,D^k$ is the disk of radius $r > 0$ centered at the origin; in addition, if Y denotes the closure of W in X, then $X - Y$ is the union of the images of the open disks of radius $\frac{1}{2}$.

Similarly, we can choose V such that

$$A \;\subset\; V \subset \; \overline{V} \;\; \subset W$$

and A is a strong deformation retract of both V and $\overline{V}$; in fact we can choose V explicitly so that $X - V$ is the union of the sets $f_j[\frac{3}{4}D^k]$ and $X - \overline{V}$ is the union of the images of the open disks of radius $\frac{3}{4}$.

Consider the following commutative diagram, in which $U \subset D^k$ denotes the set of all points v with $|v| > \frac{3}{4}$ and $C \subset D^k$ denotes the set of points v with $|v| \geq \frac{1}{2}$; the map G is defined by suitably restricting F, and the map K is also defined by F. Our hypotheses imply that G is a homeomorphism of pairs. Finally, the maps ξ, p and q are inclusions.

$$\begin{array}{ccc}
\bigoplus_j \ H_*(D^k - U, A - U) & \stackrel{\oplus^M \xi_*}{\longrightarrow} & \bigoplus_j \ H_*(D^k, C) \\
\Big\downarrow \cong & & \Big\downarrow \cong \\
H_*\left(\coprod_j \ (D^k - U, C - U)\right) & \stackrel{p_*}{\longrightarrow} & H_*\left(\coprod_j \ (D^k, C)\right) \\
\Big\downarrow G_* & & \Big\downarrow K_* \\
H_*(X - V, Y - V) & \stackrel{q_*}{\longrightarrow} & H_*(X, Y)
\end{array}$$

The maps ξ_*, p_* and q_* are excision isomorphisms, and G_* is an isomorphism because G is a homeomorphism of pairs. Therefore we can conclude that K_* is also an isomorphism.

Now consider the following commutative diagram, in which the horizontal arrows are induced by inclusions of pairs.

$$\begin{array}{ccc}
\bigoplus_j \ H_*(D^k, S^{k-1}) & \stackrel{\oplus^M \zeta_*}{\longrightarrow} & \bigoplus_j \ H_*(D^k, C) \\
\Big\downarrow \cong & & \Big\downarrow \cong \\
H_*\left(\coprod_j \ (D^k, S^{k-1})\right) & \longrightarrow & H_*\left(\coprod_j \ (D^k, C)\right) \\
\Big\downarrow F_* & & \Big\downarrow K_* \\
H_*(X, A) & \longrightarrow & H_*(X, Y)
\end{array}$$

By construction we know that A is a deformation retract of Y, so that the homology maps $H_*(A) \to H_*(Y)$ are isomorphisms. A Five Lemma argument then shows that the homology maps $H_*(X, A) \to H_*(X, Y)$ are also isomorphisms. Likewise, we know that the maps $H_*(S^{k-1}) \to H_*(C)$ are isomorphisms, so the same is true for the maps $H_*(D^k, S^{k-1}) \to H_*(D^k, C)$ and the direct sum of M copies of these maps. Therefore the upper horizontal arrow is an isomorphism. We just saw that the lower horizontal arrow is an isomorphism, and in the preceding step we saw that K_* is an isomorphism, and therefore it follows that F_* must also be an isomorphism, which is what we wanted to prove.■