Roadmap for Files on the Lefschetz Fixed Point Theorem

Sections III.3 and III.4 of **http://math.ucr.edu/~res/math246A-2012/advancednotes2012.pdf** lead to a proof of the Lefschetz Fixed Point Theorem; the first of these sections concerns a key result needed in the proof (the ***Simplicial Approximation Theorem***), and the second concerns the proof of the theorem itself. Each section has references to Hatcher, at various points other files in the course directory are needed. The purpose here is to explain the roles of the various files in logical sequence.

The whole discussion should begin with the following definitions:

Definition 1. If $(\mathbf{P},\mathbf{K})$ and $(\mathbf{Q},\mathbf{L})$ are simplicial complexes, a continuous mapping $\mathbf{g}:\mathbf{P} \to \mathbf{Q}$ is said to be ***simplicial*** with respect to $\mathbf{K}$ and $\mathbf{L}$ provided the following hold:

1. If $\mathbf{v}_0, \dots, \mathbf{v}_q$ are vertices of a simplex in $\mathbf{K}$, then $\mathbf{g}(\mathbf{v}_0), \dots, \mathbf{g}(\mathbf{v}_q)$ are vertices of a simplex in $\mathbf{L}$ (duplications in either vertex list are permitted).
2. If $\mathbf{x}$ lies on the simplex $\mathbf{v}_0 \dots \mathbf{v}_q$ in $\mathbf{K}$ and $\mathbf{x} = \sum \mathbf{t}_i \mathbf{v}_i$ expresses $\mathbf{x}$ as a convex combination of the vertices, then $\mathbf{g}(\mathbf{x}) = \sum \mathbf{t}_i\, \mathbf{g}(\mathbf{v}_i)$.

Definition 2. If $\mathbf{g}: (\mathbf{P},\mathbf{K}) \to (\mathbf{Q},\mathbf{L})$ is a simplicial mapping, the mapping $\mathbf{g}_{\#}:\mathbf{C}_*(\mathbf{K}) \to \mathbf{C}_*(\mathbf{L})$ of ***unordered*** simplicial chain complexes is given on the standard free generators $\mathbf{v}_0 \dots \mathbf{v}_q$ of $\mathbf{C}_q(\mathbf{P})$ by $\mathbf{g}_{\#}(\mathbf{v}_0 \dots \mathbf{v}_q) = \mathbf{g}(\mathbf{v}_0) \dots \mathbf{g}(\mathbf{v}_q)$. — As noted in Proposition III.3.1 of the main notes, this yields a covariant functor on simplicial maps, and the standard natural chain transformation from unordered simplicial chains to singular chains is a natural transformation from the induced simplicial chain map functor to the induced singular chain map functor, and passage to homology groups yields a natural transformation from simplicial homology to singular homology.

The Simplicial Approximation Theorem

The usefulness of the preceding constructions and Proposition III.3.1 will ultimately depend upon knowing something about approximating an arbitrary continuous map $\mathbf{f}: \mathbf{P} \to \mathbf{Q}$ by a simplicial mapping. Although it is too much to expect that there is always a simplicial mapping $\mathbf{g}: (\mathbf{P},\mathbf{K}) \to (\mathbf{Q},\mathbf{L})$ which is homotopic to $\mathbf{f}$, but always exist if we are willing to replace the decomposition $\mathbf{K}$ with some iterated barycentric subdivision. The precise statement of the Simplicial Approximation Theorem is given on pages 2 – 4 (including pages 2A and 2B) of **http://math.ucr.edu/~res/math246A-2012/footnotes2unit3.pdf**, and there is further discussion at the end of Section III.3 beginning with the paragraph preceding Proposition III.3.3. Proposition 0 on page 1 of **http://math.ucr.edu/~res/math246A-2012/footnotes2unit3.pdf** can be combined with Corollary III.3.1 to show that one can compute the singular homology group homomorphisms associated to $\mathbf{f}$ from the simplicial chain complex homomorphisms associated to $\mathbf{g}$ and the iterated barycentric subdivision chain maps from $\mathbf{C}_*(\mathbf{K})$ to $\mathbf{C}_*(\mathbf{B}^r(\mathbf{K}))$.

Rationalizations of homology theories

In order to formulate the Lefschetz Fixed Point Theorem, we need a generalization of the trace in linear algebra — which is defined for linear transformations from a finite−dimensional vector space to themselves — to endomorphisms of finitely generated abelian groups (recall that an ***endomorphism*** is a homomorphism whose domain and codomain are equal). One quick and dirty way of doing this is to define a construction which converts an abelian group $\mathbf{A}$ into a rational vector space $\mathbf{A}_{(0)}$ and converts a homomorphism $\mathbf{f}$ of abelian groups into a linear transformation $\mathbf{f}_{(0)}$ of rational vector spaces such that the assignment $\mathbf{f} \rightarrow \mathbf{f}_{(0)}$ defines a covariant functor from abelian groups to rational vector spaces. The explicit construction described in Section VII.5 of **http://math.ucr.edu/~res/math246A-2012/algtopnotes2012.pdf** is a generalization of the construction of the rational numbers from the integers, and it is a special case of a more general construction known as ***localization*** or ***forming modules of quotients***. We can then use this construction to *set the trace of* $\mathbf{f}$ *equal to the trace of* $\mathbf{f}_{(0)}$.

Theorem VII.5.2 and Corollary VII.5.3 show that rationalization sends exact sequences and homology groups to homology modules (or vector spaces since we are working over the rationals). In particular, we can apply the rationalization construction to simplical or singular homology, and the resulting vector spaces and linear transformations define homology theories valued in the category of rational vector spaces (and such that the homology of a point is the rational numbers). Some details for the proof of the first result are written up in the file **http://math.ucr.edu/~res/math246A-2012/rationalization.pdf**.

The Lefschetz Fixed Point Theorem

Some preliminaries are discussed at the beginning of Section III.4, continuing through the proof of Lemma III.4.4. At this point the next step is the statement and proof of the Lefschetz Fixed Point Theorem, and the background, statement and proof for this result are described on the last six pages of **http://math.ucr.edu/~res/math246A-2012/footnotes2unit3.pdf**. This provides everything needed to go through the remaining portion of Section III.4 (beginning with the subheading "***Vector fields on*** $\mathbf{S}^2$" on line −13 of page 69 in the main course notes file **http://math.ucr.edu/~res/math246A-2012/advancednotes2012.pdf**).

Finally, here is one more consequence of the Lefschetz Fixed Point Theorem:

COROLLARY III.4.8. *Let* $\mathbf{n} \geq \mathbf{1}$, *and let* $\mathbf{f}:\mathbf{S}^n \rightarrow \mathbf{S}^n$ *be continuous. If the degree of* $\mathbf{f}$ *is not equal to* $(-1)^{n+1}$, *then* $\mathbf{f}$ *has a fixed point.*

Proof. If $\mathbf{d}$ is the degree of $\mathbf{f}$, then the Lefschetz number of $\mathbf{f}$ is equal to $1 + (-1)^n$, and this number is zero if and only if $\mathbf{d} = (-1)^{n+1}$. ■

Biographical note. Solomon Lefschetz (1884 – 1972) was one of the foremost mathematicians of the 20th century, and the ***MacTutor*** and ***Wikipedia*** biographical articles

http://www-history.mcs.st-andrews.ac.uk/Biographies/Lefschetz.html

http://en.wikipedia.org/wiki/Solomon_Lefschetz

give very accessible accounts of his remarkable life story and research achievements.