Exercises involving Mayer-Vietoris sequences

1. (i) Suppose that we are given a long exact sequence of the form

$$\cdots \to C_{n+1} \to A_n \to B_n \to C_n \to A_{n-1} \to \cdots$$

such that each of the groups A_n and B_n is a finitely generated abelian group. Prove that each of the groups C_n is also a finitely generated abelian group. [*Hint:* What can we say about the kernel/image groups at each C_n?]

(ii) Suppose that the open set $U \subset \mathbb{R}^n$ is a union of finitely many convex open subsets. Prove by induction (on the number of subsets) that the homology groups of U are finitely generated in each dimension.

2. Let $U \subset \mathbb{R}^n$ (where $n \geq 2$) denote the complement of all points $p_k = (k, 0, \cdots, 0)$, where k runs through all nonnegative integers. Explain why $H_{n-1}(U)$ is a free abelian group on a countably infinite set of generators, and explain why this implies that U is not a union of finitely many convex open subsets. [*Hint:* Use excision to compute $H_*(\mathbb{R}^n, U)$, taking V to be the union of the open disks of radii $\frac{1}{3}$ centered at the points p_k. Compare the argument in `openRn.pdf`.]

3. If U is an open subset of $\mathbb{R}^n$ explain why $H_q(U) = 0$ if $q \geq n+1$ (in fact this is also true when $q = n$, but it fails if $k < n$ for $U = \left(\mathbb{R}^k - \{\mathbf{0}\}\right) \times \mathbb{R}^{n-k}$). [*Hint:* Let L_k be the union of all n-dimensional hypercubes in U whose vertices have coordinates of the form $m/2^k$ for some integer L_k. Why do the homology groups of L_k vanish in dimensions $\geq n+1$, and why does every compact subset of U lie in some subset L_k?]

4. Let X be a topological space, and let k be a nonnegative integer. Prove the identity

$$H_q(X \times S^k) \;\cong\; H_q(X) \;\oplus\; H_{q-k}(X) \qquad \text{(all } q\text{)}$$

by induction on k. [*Hints:* Why is this true if $k = 0$? Assume the result is true for k. Let $U_+, U_- \subset S^{k+1}$ be the complements of the north and south poles, and consider the long exact Mayer-Vietoris sequence for the decomposition

$$X \times S^{k+1} \;=\; X \times U_+ \;\cup\; X \times U_- \;.$$

You should be able to draw some conclusion about the homology of $X \times U_+ \cap X \times U_-$ from the induction hypothesis.]