# Simplicial approximation

Prop. 0  $K$ = simplicial complex, $P$ = underlying space, $\lambda : C_*(K) \longrightarrow S_*(P)$ the map previously called $\theta$. Then the simplicial and singular barycentric subdivision maps are related by the following commutative diagram:

$$\begin{array}{ccc} C_*(K) & \xrightarrow{\beta} & C_*(BK) \\ \lambda \downarrow & & \downarrow \lambda \\ S_*(P) & \xrightarrow{\beta} & S_*(P). \end{array}$$

This follows from the constructions of the maps.

<u>Simplicial Approximation Thm.</u>

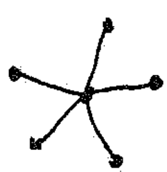
Let $(P, K)$ and $(Q, L)$ be simplicial complexes and let $f: P \to Q$ be continuous. Then $\exists r > 0$ and a simplicial map $g: B^r(K) \to L$ such that for all $x \in P$, if $\sigma$ is a minimal simplex containing $x$, then $f(x) \in g[\sigma]$.

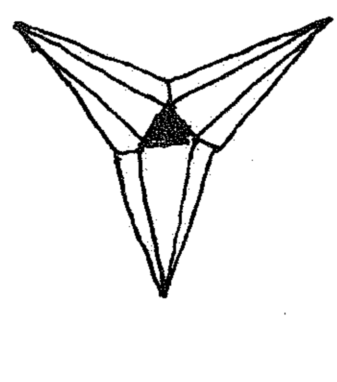
<u>NEED SOME TERMINOLOGY</u>

To some extend this corrects the definitions etc. on page 178 of Hatcher.

$v$ vertex, $\sigma$ simplex in $K$

The <u>(closed) star</u> $\operatorname{Star}\sigma$ = all simplices $\tau$ in $K$ such that $\sigma \cap \tau \neq \phi$



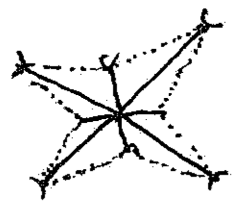


The <u>open simplex</u> $\mathring{\sigma} = \sigma - \partial\sigma$

[If $\sigma$ has vertices $v_i$, then $\mathring{\sigma}$ = all $\sum t_i v_i$ s.t. $t_i > 0\ \forall i$.

<u>Note</u> Every point lies in a unique open simplex.

The <u>open star</u> of a vertex $v$ = all open simplices $\mathring{\sigma}$ s.t. $v \in \sigma$. $st(v)$ or Open star$(v)$



($v$ = vertex in the middle)

<u>Claim</u> Open star $(v)$ is open in $|K|$ = underlying space of $K$.

In fact, $|K|$ - Open star$(v)$ is the union of all (closed) simplices $\tau$ such that $v \notin \tau$.

<u>Proof</u>: Call the described set F. Then $x \in F \Rightarrow x \in \tau$ where $v \notin \tau$. Let $\tau' \subseteq \tau$ such that $x \in \mathring{\tau}'$. Then $v$ is also not a vertex of $\tau'$, so $x \notin \text{Openstar}(v)$. Conversely, $x \notin \text{Openstar}(v) \Rightarrow x \in \mathring{\tau}$ where $v$ is not a vertex of $\tau \Rightarrow x \in \tau \subseteq F$.

<u>Key Lemma</u> $v_0, \ldots, v_q$ are vertices of a simplex in $K \Leftrightarrow \bigcap_i \text{Openstar}(v_i) \neq \phi$.

<u>Proof</u> ($\Rightarrow$) Let $v_0, \ldots, v_q$ be the vertices of $\sigma$, and let $y \in \mathring{\sigma}$. Then $y \in \cap \text{Openstar}(v_i)$ and in fact $\mathring{\sigma} \subseteq$ intersection.

($\Leftarrow$) Suppose $y \in \bigcap_i \text{openstar}(v_i)$ and let $\sigma$ be the unique simplex such that $y \in \mathring{\sigma}$. Then for each $i$, $v_i$ must be a vertex of $\sigma$, so there is a face of $\sigma$ with vertices $v_i$ (in fact, it's $\sigma$, but we don't need this).

<u>NOTE</u> In the lemma, duplications of $v_i$'s are allowed.

<u>Proof of Thm.</u> Let $\mathcal{U}_0$ be the open covering of $Q$ by sets Open star $(w)$ where $w$ runs through the vertices of $L$, and let $\mathcal{U} = f^{-1}\mathcal{U}_0$. Using Lebesgue #s and barycentric subdivision, can find some $B^r K$ s.t. (i) each subset of $P$ with diam $< \varepsilon$ lies in an element of $\mathcal{U}$, (ii) all simplices of $B^r K$ have diameter less than $\varepsilon/3$. Then each Star$(v)$ has diam $\leq 2\varepsilon/3 < \varepsilon$, so $\exists$ vertex $g(v)$ in $L$ s.t. $f[\text{Star}(v)] \subseteq$ Open star $g(v)$.

Define $g$ : Vertices of $B^r K \longrightarrow$ Vertices of $L$ using these choices.

CLAIM If $x \in v_0 \ldots v_q$, where the latter is minimal, then $f(x) \in$ simplex with vertices $g(v_0) \ldots g(v_q)$. $[x = \sum t_i v_i$, all $t_i > 0]$

This follows because $x \in \cap$ Open star$(v_i)$, so $f(x) \in \bigcap_i f[\text{Openstar}(v_i)] \subseteq \bigcap_i$ Openstar $g(v_i)$. This shows $g(v_i)$ are the vertices of a simplex, and the latter contains $f(x)$. Let $g(x) = \sum t_i g(v_i)$.

It follows that the image of the straight line homotopy

$$H(x,t) = t g(x) + (1-t) f(x)$$

lies in $P$ so the latter defines a homotopy from $f$ to $g$.

# The Lefschetz Fixed Point Theorem

$(P, K)$ polyhedron

$f : P \to P$ cont.

The Lefschetz number $\Lambda(f) =$

$$\sum (-1)^k \text{ trace } f_{k*} : H_k(P;\mathbb{Q}) \to H_k(P;\mathbb{Q}).$$

Claim This is an integer.

Idea Look at the diagram

$$\begin{array}{ccc} H_k(P;\mathbb{Z})/\text{torsion} & \xrightarrow{\subseteq} & H_k(P;\mathbb{Q}) \\ \downarrow f_{k*}^{\mathbb{Z}} & & \downarrow f_{k*}^{\mathbb{Q}} \\ H_k(P;\mathbb{Z}/\text{torsion}) & \xrightarrow{\subseteq} & H_k(P;\mathbb{Q}) \end{array}$$

and choose free generators for $H_k(P;\mathbb{Z})/\text{torsion}$ which yield a basis for $H_k(P;\mathbb{Q})$. This shows that a matrix for $f_*^{\mathbb{Q}}$ comes from an integral matrix for $f_*^{\mathbb{Z}}$.

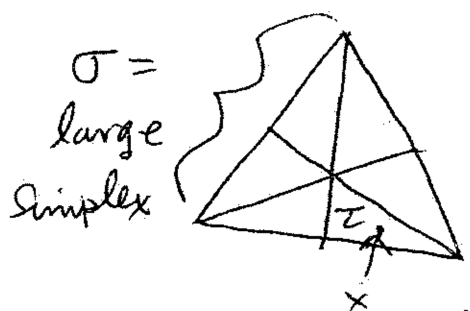
Hence each trace $f_{k*}^{\mathbb{Q}}$ is an integer.

THEOREM $\Lambda(f) \neq 0 \Rightarrow f$ has a fixed point.

PROOF. Suppose not, and let $\delta$ = min distance from $f(x)$ to $x$, so that $\delta > 0$ by compactness. Subdivide $K$ into simplices of diameter $< \delta/4$. Then $f[\sigma] \cap \sigma$ is empty for all $\sigma$, and more generally $f[\mathrm{Star}\,\sigma] \cap \mathrm{Star}\,\sigma = \phi$ for all $\sigma$. Choose a simplicial approximation $g: B^r K \to K$ to $f$ as in the Simplicial Approximation Thm.

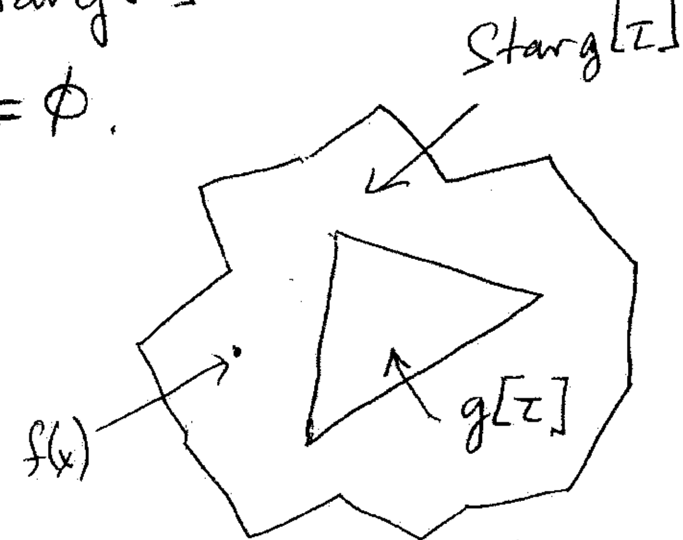
Then if $\tau$ is a simplex of $B^r(K)$ we have $f[\tau] \subseteq \mathrm{Star}\, g[\tau]$.

<u>CLAIM</u> $g[\tau] \cap \tau = \phi$.




$\tau$ in $B^r(K)$

$\tau \subseteq \sigma$ in $K$.




$g[\tau]$ in $K$.

$$x \in \tau \Rightarrow f(x) \in \mathrm{Star}\, g[\tau]$$

$$\text{and } d(x, f(x)) \geq \delta$$

$$y \in \tau \Rightarrow d(x,y) < \delta/4$$

$$z \in g[\tau] \Rightarrow f(x), z \in \mathrm{Star}\, g(\tau)$$

$$\Rightarrow d(f(x), z) < \delta/2.$$

It follows that $y \in \tau$, $z \in g[\tau] \Rightarrow$ $d(y,z) > \delta/4$.

In fact, we can say more: Given $\tau \subseteq \sigma$ we have $\sigma \cap g(\tau) = \phi$.
in $B^r(K)$   in $K$

Consider what this means for the chain map $C_*(K) \xrightarrow{\beta_r} C_*(B^r(K)) \xrightarrow{g\#} C_*(K)$:

Let $\sigma = v_0 \ldots v_q \in C_q(K)$ be a typical generator, so that $\beta_r(v_0 \ldots v_q)$ lies in the chain subcomplex $C_*(B^r(v_0 \ldots v_q))$, and consider the effect of $g_\#$ on a typical free generator of $C_q(B^r(v_0 \ldots v_q))$.

If $\sigma$ is the simplex with vertices $v_0 \ldots v_q$, then $g_\#$ must take a typical free generator of $C_q(B^r(v_0 \ldots v_q))$ into a chain subcomplex $C_*(\sigma') \subseteq C_*(K)$ such that $\sigma' \cap \sigma = \phi$. Therefore the image

of $v_0 \ldots v_q$ in $C_q(K)$ actually lies in some $C_q(K')$ where $\sigma$ and $K'$ are disjoint, and hence $g_\# \beta_r(v_0 \ldots v_q)$ will lie in a subgroup of $C_q(K)$ consisting of chains whose $v_0 \ldots v_q$-coordinates are zero. This means that the trace of $g_\# \beta_r : C_q(K) \longrightarrow C_q(K)$ is zero, and by the trace identity the same is true for $H_q(K;\mathbb{Q}) \xrightarrow{\beta_{r*}} H_q(B^r(K);\mathbb{Q}) \xrightarrow{g_*} H_q(K;\mathbb{Q})$.
We know the latter corresponds to the singular homology map

$$g_* : H_q(P;\mathbb{Q}) \longrightarrow H_q(P;\mathbb{Q})$$

and hence its trace is also zero.
Taking alternating sums, we see

that $\Lambda(g)=0$. Finally, $f \simeq g \Rightarrow \Lambda(f)=\Lambda(g)$, and hence we also have $\Lambda(f)=0$.

To summarize, we have shown that if $f$ has no fixed points, then $\Lambda(f)=0$.