page ~~128~~ 123, lines $-15$ to $-14$

Verification that the permutation $\tau_{p,q}$

$$\begin{array}{c} 1 \ldots p \,\vdots\, p+1 \ldots p+q \\ \downarrow \\ p+1 \ldots p+q \,\vdots\, 1 \ldots p \end{array}$$

has sign $(-1)^{pq}$.

Follow the suggestion on line $-14$, fixing $p$ and proceeding by induction on $q$.

$\underline{q=1}$ The permutation is just the cycle $(1 \ldots p\ p+1)$, which has sign $(-1)^p$.

Assume for $q=k$, and suppose $q=k+1$.

Then $\operatorname{sgn} \tau_{p,k} = (-1)^{pk}$ and $\tau_{p,k}$ is

$$\begin{array}{c} 1 \ldots p \,\vdots\, p+1 \ldots p+k \,\vdots\, p+k+1 \\ \downarrow \\ p+1 \ldots p+k \,\vdots\, 1 \ldots p \,\vdots\, p+k+1 \end{array}$$

The latter implies $\tau_{p,k+1} = (k+1 \ldots p+k+1) \circ \tau_{p,k}$ and hence has sign $= (-1)^p \cdot (-1)^{pk} = (-1)^{p(k+1)}$ completing the verification of the inductive step.

page ~~123~~, lines -11 to -10

Verification of Proposition 4.

By construction there is a commutative diagram ($\Omega$ = quotient projection)

$$\begin{array}{ccc} Cov_p(U) \times Cov_q(U) & \xrightarrow{\otimes} & Cov_{p+q}(U) \\ \downarrow \Omega\times\Omega & & \downarrow \Omega \\ \Lambda^p(U) \times \Lambda^q(U) & \xrightarrow{\wedge} & \Lambda^{p+q}(U) \end{array}$$

for all $p$ and $q$, so if $\overline{\lambda}, \overline{\omega}, \overline{\theta} \in Cov_*(U)$ project to $\lambda, \omega, \theta$ we have

$$(\theta \wedge \omega) \wedge \lambda = (\Omega\overline{\theta} \wedge \Omega\overline{\omega}) \wedge \Omega\overline{\lambda} =$$

$$\Omega(\overline{\theta} \otimes \overline{\omega}) \wedge \Omega(\overline{\lambda}) = \Omega((\overline{\theta} \otimes \overline{\omega}) \otimes \overline{\lambda}) \xrightarrow[\text{associative}]{\text{since } \otimes \text{ is}}$$

$$\Omega(\overline{\theta} \otimes (\overline{\omega} \otimes \overline{\lambda})) = \Omega(\overline{\theta}) \wedge \Omega(\overline{\omega} \otimes \overline{\lambda}) =$$

$$\Omega(\overline{\theta}) \wedge (\Omega\overline{\omega} \wedge \Omega\overline{\lambda}) = \theta \wedge (\omega \wedge \lambda).$$

page ~~124~~ 125, lines 1-6

Proof of Theorem 5

(i) Since both the left and right hand sides are $\mathbb{R}$-bilinear with respect to $\theta$ and $\omega$, it suffices to verify this result for sets of forms $\theta_\alpha$ and $\omega_\alpha$ which span $\Lambda^p(U)$ and $\Lambda^q(U)$. The obvious choices are forms given by $f\, dx^{i_1} \wedge \dots \wedge dx^{i_p}$ and $g\, dx^{j_1} \wedge \dots \wedge dx^{j_q}$ where $f, g \in C^\infty(U)$.

$\boxed{\text{PRELIMINARY Note}}$

The relationship

$$\underset{x^i}{\text{exterior derivative of coord. fcn}} = \text{"}dx^i\text{"}$$

looks tautological, but it isn't quite. One has $d(x^i) = \sum \frac{\partial x^i}{\partial x^j} \text{"}dx^j\text{"}$ which reduces to "$dx^i$" because $\frac{\partial x^i}{\partial x^j} = \delta_{ij}$ (Kronecker delta).

page ~~124~~ 12, lines 1-6 continued

So we need to compare

$$d((f\,dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge(g\,dx^{j_1}\wedge\cdots\wedge dx^{j_q})$$ and

$$d(f\,dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge(g\,dx^{j_1}\wedge\cdots\wedge dx^{j_q}) +$$
$$(-1)^p(f\,dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge d(g\,dx^{j_1}\wedge\cdots\wedge dx^{j_q}).$$

The first of these is

$$d(fg)\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}\wedge dx^{j_1}\cdots dx^{j_q} =$$
$$\sum f\frac{\partial g}{\partial x^k}dx^k\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q} +$$
$$\sum \frac{\partial f}{\partial x^l}g\,dx^l\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q} =$$
$$(-1)^p(f\,dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge(dg\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q}) +$$
$$(df\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge(g\,dx^{j_1}\wedge\cdots\wedge dx^{j_q}) =$$

$$(-1)^p\,\theta\wedge d\omega + d\theta\wedge\omega.$$

<u>page 12~~4~~, lines 1-6 continued</u>

(ii) Since $d \circ d$ is $\mathbb{R}$-linear, as in (i) it suffices to prove it for a form of type $f\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$.

It is routine to check that $ddx^i = 0$ for all $i$, and hence by (i) we have $d(dx^{i_1} \wedge \cdots \wedge dx^{i_p}) = 0$.

Now $df = \sum_k \frac{\partial f}{\partial x^k} dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ and

$$ddf = \sum_{j,k} \frac{\partial^2 f}{\partial x^j \partial x^k} dx^j \wedge dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} =$$

$$\sum_{j<k} + \sum_{j=k} + \sum_{j>k}.$$ The middle term vanishes because $dx^j \wedge dx^j = 0$, and the first and third cancel because

$$\frac{\partial^2 f}{\partial x^j \partial x^k} dx^j \wedge dx^k = -\frac{\partial^2 f}{\partial x^k \partial x^j} dx^k \wedge dx^j \quad \text{if } j \neq k.$$

page 125 lines 10-12

(i) In this case we have an isomorphism of free $C^\infty(U)$ modules

$$\Phi_1 : \text{Vector Fields} \longrightarrow 1\text{-forms}$$

with $\Phi_1$ of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ equal to $dx, dy, dz$.

Then $\nabla f = \frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial}{\partial z}$, and

$$\Phi_1(\nabla f) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

(ii) In this case $\Phi_2$ sends $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ to $dy\wedge dz$, $dz\wedge dx$ [NOTE!!], $dx\wedge dy$ and

$$d\,\Phi_1\left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}\right) = d(P\,dx + Q\,dy + R\,dz)$$

$$= \frac{\partial P}{\partial y}dy\wedge dx + \frac{\partial P}{\partial z}dz\wedge dx + \frac{\partial Q}{\partial x}dx\wedge dy + \frac{\partial Q}{\partial z}dz\wedge dy + \frac{\partial R}{\partial x}dx\wedge dz + \frac{\partial R}{\partial y}dy\wedge dz = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dy\wedge dz +$$

$$\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dz\wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\wedge dy$$

<u>page 125, lines 10-12 continued</u>

and the latter is equal to

$$\Phi_2\Big(\underbrace{\nabla\times}_{\text{CURL}}\Big(P\frac{\partial}{\partial x}+Q\frac{\partial}{\partial y}+R\frac{\partial}{\partial z}\Big)\Big)$$

(iii) $d\,\Phi_2\left(P\frac{\partial}{\partial x}+Q\frac{\partial}{\partial y}+R\frac{\partial}{\partial z}\right)=$

$d(P\,dy\wedge dz+Q\,dz\wedge dx+R\,dx\wedge dy)=$

$\frac{\partial P}{\partial x}dx\wedge dy\wedge dz+\frac{\partial Q}{\partial y}dy\wedge dz\wedge dx+\frac{\partial R}{\partial z}dz\wedge dx\wedge dy$

$=\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\wedge dy\wedge dz=$

$$\Phi_3\Big(\underbrace{\nabla\cdot}_{\text{DIV}}\Big(P\frac{\partial}{\partial x}+Q\frac{\partial}{\partial y}+R\frac{\partial}{\partial z}\Big)\Big).$$

page 138, lines 5-7

The decomposition $U = \cup K_m$, where

(i) $U \subseteq \mathbb{R}^n$ is open

(ii) $K_m$ is compact and $K_m \subseteq \mathrm{Int}\, K_{m+1}$

is actually a special case of the following purely point-set theoretic result.

Proposition Let $X$ be locally compact $T_2$ and second countable. Then $X = \cup K_m$, where $K_m$ is compact and $K_m \subseteq \mathrm{Int}\, K_{m+1}$.

Proof ① Claim $X$ has a countable base of open sets $U_k$ such that $\overline{U_k}$ is compact.

Let $\mathfrak{B} = \{V_p\}_{p \geq 0}$ be a base, and let $x \in X$, so $x \in V_{p_0}$ some $p_0$. Then there is an open set $W$ s.t. $x \in W \subseteq \overline{W} \subseteq V_{p_0}$ and $\overline{W}$ is compact. Since $\mathfrak{B}$ is a base for $X$, we know that

$W = \bigcup_{j \in J} V_j$ for some $J \subseteq \mathbb{N}$, and each $\overline{V_j}$ is compact since $\overline{V_j} \subseteq \overline{W}$ and the latter is compact. Hence the set $\mathfrak{V}'$ of all $V_q \in \mathfrak{V}$ with $\overline{V_q}$ compact is also a base for $X$.

(2) By the preceding, there is a countable base $\mathfrak{U} = \{U_\ell\}_{\ell \geq 0}$ with compact closures. Pick $x_0 \in X$, and let $x_0 \in U_{k_0}$, and set $K_0 = \overline{U_{k_0} \cup U_0}$. Given $K_m$, construct $K_{m+1}$ as follows: One can find a finite collection of open sets $\{U_j\}_{j \in J \subseteq \mathbb{N}}$ in $\mathfrak{U}$ such that $K_m \subseteq \bigcup_{j \in J} U_j$. Let $K_{m+1}$ be the closure of $\left(\bigcup_{j \in J} U_j\right) \cup U_{m+1}$. Clearly $K_m \subseteq \operatorname{Int} K_{m+1}$, $K_{m+1}$ is also compact, and $X = \bigcup_{m \in \mathbb{N}} U_m \subseteq \bigcup K_m \subseteq X$ shows $X = \bigcup_m K_m$.

page 138, lines -12 to -5

Direct products of cochain complexes

Let $\{(C_\alpha, \delta_\alpha)\}$ be an indexed family of cochain complexes. Then $(\prod C_\alpha, \prod \delta_\alpha)$ is a chain complex because $(\prod \delta_\alpha)^2 \xrightarrow[\text{of direct product}]{\text{functoriality}}$

$$\prod \delta_\alpha^2 = \prod 0 = 0.$$

Claim The projection maps

$\prod_\alpha H^*(C_\alpha) \to H^*(C_\gamma)$ induce an isomorphism
some $\gamma$

$$H^*\left(\prod_\alpha C_\alpha\right) \to \prod_\alpha H^*(C_\alpha).$$

One needs to check that $(x_\alpha) \in \prod C_\alpha$ is a cocycle/coboundary $\Leftrightarrow$ each $x_\gamma$ is,

and then one can use the fact that if $A_\alpha$ is a submodule of $B_\alpha$ for all $\alpha$, then

$$\prod_\alpha (A_\alpha / B_\alpha) \cong \left[\prod_\alpha A_\alpha\right] / \left[\prod_\alpha B_\alpha\right].$$