## Complements of finite sets in $\mathbb{R}^n$

We shall begin by stating the two main results.

**THEOREM 1.** *If $X$ is a finite set with $k \geq 1$ elements and $n \geq 3$. then $\mathbb{R}^n - X$ is simply connected.*

**THEOREM 2.** *Let $n \geq 2$, let $S \subset \mathbb{R}^n$ consist of exactly $p$ points, and let $T \subset \mathbb{R}^n$ consist of exactly $q$ points. If $p \neq q$, then $\mathbb{R}^n - S$ and $\mathbb{R}^n - T$ are not homeomorphic.*

The first result will be proved using the Seifert-van Kampen Theorem. On the other hand, we shall prove Theorem 2 using techniques from 205B, most notably homology theory, referring to the file of notes for that course

`http://math.ucr.edu/~res/math205B-2012/algtop-notes.pdf`

freely and whenever it is necessary.

**Proof of Theorem 1.** If $X$ contains only one point, then the space $\mathbb{R}^n - X$ is homeomorphic to $S^{n-1} \times (0, \infty)$, and therefore the simple connectivity of $\mathbb{R}^n - X$ follows from the simple connectivity of $S^{n-1}$.

*The inductive step.* Suppose that the result is true for $\mathbb{R}^n - Y$ if $Y$ contains strictly less than $k \geq 2$ points, and suppose that $X$ contains exactly $k$ points. For each $j$ such that $1 \leq j \leq n$ let $M_j$ be the largest $j^{\text{th}}$ coordinate among all the points of $X$. Since there cannnot be two points $x$ and $y$ such that $x_j = M_j = y_j$ for all $j$, we can find a pair of points $u$ and $v$ and some $j(0)$ such that $u_{j(0)} < v_{j(0)}$. Furthermore, since there are only finitely many possibilities for the value of this coordinate for points in the finite set $X$, we can choose $u$ so that it has the largest $j(0)^{\text{th}}$ coordinate among all points whose $j(0)^{\text{th}}$ coordinate is strictly less than $v_{j(0)}$.

Choose $c_1, c_2 \in \mathbb{R}$ such that $u_{j(0)} < c_1 < c_2 < v_{j(0)}$, and let $U_2$ and $U_1$ be the sets of points in $\mathbb{R}^n$ such that $x_{j(0)} < c_2$ and $x_{j(0)} > c_1$ respectively. Then both $U_2 \cap X$ and $U_1 \cap X$ are nonempty but $U_1 \cap U_2 \cap X$ is empty; let $p_1, p_2 > 0$ be such that $|U_i \cap X| = p_i$. Consider the decomposition $\mathbb{R}^n - X = V_1 \cup V_2$, where $V_i = U_i - (U_i \cap X)$ is arcwise connected and $V_1 \cap V_2 = U_1 \cap V_2$. Since each of the sets $U_1$, $U_2$ and $U_1 \cap U_2$ is homeomorphic to $\mathbb{R}^n$ (for example, the intersection is homeomorphic to $(c_1, c_2) \times \mathbb{R}^{n-1}$ and the latter is homeomorphic to $\mathbb{R}^n$) we have can apply the Seifert-van Kampen Theorem to conclude that $\pi_1(\mathbb{R}^n - X)$ is isomorphic to a free product of $\pi_1(\mathbb{R}^n - Y_1)$ and $\pi_1(\mathbb{R}^n - Y_2)$ where $Y_i$ has $p_i$ elements for $i = 1, 2$. By induction the fundamental groups of the spaces $\mathbb{R}^n - Y_i$ are trivial, and therefore the fundamental group of $\mathbb{R}^n - X$ is also trivial.■

**Proof of Theorem 2.** Since homology groups of homeomorphic spaces are isomorphis, it will suffice to prove the following assertion:

> *If $n \geq 2$ and $X \subset \mathbb{R}^n$ consists of $k$ points, then the singular homology groups of $\mathbb{R}^n - X$ are given by $H_j(\mathbb{R}^n - X) \cong \mathbb{Z}$ if $k = 0$, $H_j(\mathbb{R}^n - X) \cong \mathbb{Z}^k$ if $k = n-1$, and $H_j(\mathbb{R}^n - X) \cong 0$ otherwise.*

It follows that if $X'$ has $k'$ points and $k \neq k'$, then the complements of $X$ and $X'$ are not isomorphic so these spaces cannot even be homotopy equivalent.

To prove the assertion, for each $x \in X$ let $U_x$ be the open neighborhood of radius $r$ centered at $x$; choose $r$ to be smaller than half the minimum distance between points of $X$ (the minimum exists by the finiteness of $X$), and let $U = \cup_x U_x$, so that $\mathbb{R}^n = U \cup (\mathbb{R}^n - X))$ and $U \cap X = \cup_x U_x - \{x\}$.

Then by excision, the splitting of the homology of $X$ into the homology of its arc componenents, and Theorem VII.1.7 in `algtop-notes.pdf` we know that

$$H_j(\mathbb{R}^n, \mathbb{R}^n - X) \quad \cong \quad H_j(U, U - X) \quad \cong \quad H_j\left(\cup_x U_x, \cup_x U_x - \{x\}\right) \quad \cong$$

$$\bigoplus_x H_j\left(\, U_x,\, U_x - \{x\}\right) \quad \cong \quad \mathbb{Z}^k \quad \text{or} \quad 0$$

where the group is zero unless $j = n - 1$. We can now recover the homology groups $\mathbb{R}^n - X$ from the long exact homology sequence for $(\mathbb{R}^n, \mathbb{R}^n - X)$ and the fact that $H_j(\mathbb{R}^n)$ is $\mathbb{Z}$ if $j = 0$ and zero otherwise.■

Finally, we note that the argument proving Theorem 1 can also be used to prove the following result:

**PROPOSITION 3.** *If $X$ is a finite set with $k \geq 1$ elements, then the fundamental group of $\mathbb{R}^2 - X$ is a free group on $k$ generators.*

A straightforward modification of the proof of Theorem 1 yields this result; the details are left to the reader as an exercise.■